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Chern Classes

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Introduction

Chern classes were first introduced by Shiing-Shen Chern in 1946 with his work about Characteristic Classes of Hermitian Manifolds [2]. They are characteristic classes associated to complex vector bundles which allow to classify them. If two vector bundles do not have the same Chern classes, then they are different. However, the converse is not true. The aim of this paper is to introduce Chern classes and to show how one computes with this tool. Most applications will be done on the complex projective space of dimension n. Some knowledge in homological algebra and in category theory is of help in order to fully understand this paper. In the appendix the reader may find some essential backgrounds on category theory. The thesis is structured as follows.
The first chapter is a preliminary one which should allow the reader to familiarize himself with the notion of cohomology ring $H^*(X) = H^*(X, \mathbb{Z})$ associated to a topological space X. This is needed in order to introduce Chern classes. In particular we consider the simplicial homology theory and comment on the computations of the cohomology ring of $\mathbb{CP}^n$.
The second chapter is the main body of this paper and it is entirely devoted to Chern classes and some related concepts such as the Chern character or the Todd class. Two indispensable results around this subject are the *Leray-Hirsch* Theorem and the splitting principle. The chapter contains some nice computations and especially a useful recursive formula generating each term of the Chern character. It turns out that they are nothing else than Newton's identities. We will also express the Chern character and the Todd class of a given complex vector bundle in terms of its different Chern classes. Finally, we will end this paper by giving some appropriate applications to the n-dimensional complex projective space.

I take this opportunity to express gratitude to my thesis supervisor Dr. Oleksandr Iena for giving me the chance to work on this subject and I would like to thank him also very much for his precious collaboration and time he invested in this project.

Chapter 1

Preliminaries

The aim of this chapter is to introduce for a topological space X the so called **cohomology ring** $H^*(X) = H^*(X, \mathbb{Z})$. For this we will proceed as follows: In a first step we will try to understand the nature of $H^*(X)$ as an abelian group where it is just the direct sum of all cohomology groups

$$H^*(X) = \bigoplus_i H^i(X, \mathbb{Z}).$$

Before, we will familiarize ourselves with homology and homology groups in order to give a meaning to cohomology groups. In this paper we will use the approach that one can define cohomology groups on the basis of homology groups. In fact, the Hom functor allows us to switch from homology groups to cohomology groups. Below the reader will find some details of this approach.
Once we have understood $H^*(X)$ as an abelian group, a construction called the **cup product** provides the multiplication in $H^*(X)$, which respects the grading and turns $H^*(X)$ into a so-called graded-commutative ring.

1.1 Homology

We will work here over the category of abelian groups and consider homology groups with coefficients in $\mathbb{Z}$.

Definition 1.1.1 (**Chain complex**). A sequence

$$\cdots \to C_{n+1} \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \to \cdots$$

of abelian groups C_n, $n \in \mathbb{Z}$, connected by homomorphisms
$\delta_n : C_n \to C_{n-1}$ such that the composition of any two consecutive maps is zero (*i.e.* $\delta_n \circ \delta_{n+1} = 0$ for all n) is called a **chain complex**. We will denote such a sequence by $(C_\bullet, \delta_\bullet)$.

Definition 1.1.2. The elements of C_n are the **chains** (resp. the chains of degree n), the elements of $\text{Ker } \delta_n \subseteq C_n$ are the **cycles**, and those of $\text{Im } \delta_{n+1} \subseteq C_n$ are referred to as **boundaries**. The operator δ is said to be a **boundary operator** or a **homology operator**.
This vocabulary originates from topological context. The Example of homology theory given in the next section will help to see the link with this terminology.

Remark 1.1.3. The map $\delta_n : C_n \to C_{n-1}$ decreases the chain degree by 1. We then say that δ has degree -1.
The condition $\delta_n \circ \delta_{n+1} = 0$, for all n, means exactly that $\text{Im } \delta_{n+1} \subseteq \text{Ker } \delta_n$, so that the quotient space $\text{Ker } \delta_n / \text{Im } \delta_{n+1}$ makes sense.

Definition 1.1.4. We call the quotient group

$$H_n(C, \mathbb{Z}) := {}^{\text{Ker } \delta_n}\!/_{\text{Im } \delta_{n+1}}$$

the n**-th homology group** of the chain complex $(C_\bullet, \delta_\bullet)$. The direct sum

$$H_*(C, \mathbb{Z}) := \bigoplus_n H_n(C, \delta)$$

is referred to as the **homology group** of $(C_\bullet, \delta_\bullet)$.

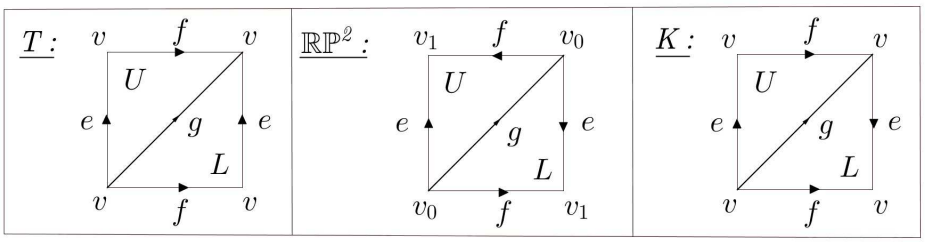
Definition 1.1.5. A sequence such that $\text{Ker } \delta_n = \text{Im } \delta_{n+1}$, at any spot n, *i.e.* a sequence whose n-th homology group $H_n(C, \mathbb{Z})$ vanishes at each spot n, is referred to as an **exact sequence**.

In the following subsection we will introduce an example of a homology theory called **simplicial homology**.

1.1.1 Simplicial Homology

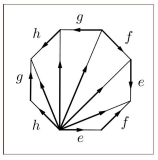
One of the most important homology theories in algebraic topology is called singular homology. Since the technical apparatus of singular homology is somewhat complicated and since we only want to give an exemplification of a homology theory, we will introduce a more primitive version called **simplicial homology** in order to see how some of the apparatus works in a simpler setting. The reader who is interested in singular homology can have a look at [7].
The natural domain of definition for simplicial homology is a class of spaces we call Δ**-complexes**, which we will introduce right now.

The torus, the projective plane, and the Klein bottle can each be obtained from a square by identifying opposite edges in the way indicated by the arrows in the following figures:



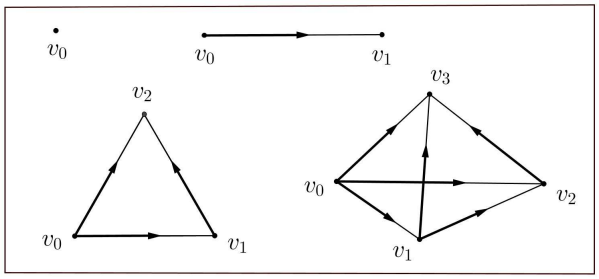
Torus Projective Plane Klein Bottle

Cutting a square along a diagonal produces two triangles, so each of these surfaces can also be built from two triangles by identifying their edges in pair. In similar fashion a polygon with any number of sides can be cut along diagonals into triangles, so in fact all closed surfaces can be constructed from triangles by identifying edges. Thus we have a single building block, the triangle, from which all surfaces can be constructed. Using only triangles we could also construct a large class of 2-dimensional spaces that are not surfaces in the strict sense, by allowing more than two edges to be identified together at a time.



The idea of a Δ-complex is to generalize constructions like these to any number of dimensions.

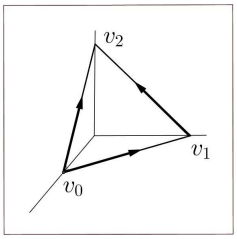
Definition 1.1.6 (**n-simplex**). The n-dimensional analog of the triangle is the **n-simplex**. This is the convex hull in a Euclidean space $\mathbb{R}^m$ $(m > n)$ of a fixed tuple of $n+1$ points $v_0, \ldots, v_n$ that do not lie in a hyperplane of dimension less than n, where by a hyperplane we mean the set of solutions of a system of linear equations. An equivalent condition would be that the difference vectors $v_1 - v_0, \ldots, v_n - v_0$ are linearly independent. The points v_i are the **vertices** of the simplex, and the simplex itself is denoted by $[v_0, \ldots, v_n]$.



Examples for n-simplices, $n = 0, 1, 2, 3$

Example 1.1.7 (**The standard n-simplex**). The **standard n-simplex** whose vertices are the unit vectors along the coordinate axes is given by the equation

$$\Delta_n = \left\{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \textstyle\sum_i t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i\right\}.$$



The standard 2-simplex

Remark 1.1.8. For purposes of homology it will be important to keep track of the order of the vertices of a simplex, so 'n-simplex' will really mean 'n-simplex with an ordering of its vertices'. A consequence of ordering the vertices of a simplex $[v_0, \ldots, v_n]$ is that this determines orientations of the edges $[v_i, v_j]$ according to increasing subscripts, as one can see in the two figures given previously.

Definition 1.1.9.

- If we delete one of the $n+1$ vertices of an n-simplex $[v_0, \ldots, v_n]$, then the remaining n vertices span an $(n-1)$-simplex, called a **face** of $[v_0, \ldots, v_n]$. Moreover, considering the ordering we will adopt the following convention:
  The vertices of a face, or of any subsimplex spanned by a subset of the vertices, will always be ordered according to their order in the larger simplex.

- The union of all the faces of Δ_n is the **boundary** of Δ_n, written $\delta\Delta_n$. The **open simplex** $\mathring{\Delta}_n$ is $\Delta_n - \delta\Delta_n$, the interior of Δ_n. By convention we set that $\mathring{\Delta}_0 := \Delta_0$.

We are now ready to give the general definition of a **Δ-complex** structure on a topological space X:

Definition 1.1.10 (**Δ-complex**). A **Δ-complex** structure on a topological space X is a collection of maps $\sigma_\alpha : \Delta_n \to X$ called characteristic maps, with n depending on the index α such that:

i) The restriction $\sigma_\alpha|_{\mathring{\Delta}_n}$ is injective, and each point of X is in the image of exactly one such restriction $\sigma_\alpha|_{\mathring{\Delta}_n}$.

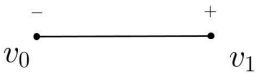
ii) Each restriction of σ_α to a face of Δ_n is one of the maps $\sigma_\beta : \Delta_{n-1} \to X$. Here we are identifying the face of Δ_n with Δ_{n-1} by the canonical linear homeomorphism between them that preserves the ordering of the vertices.

iii) A set $A \subset X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in Δ_n for each σ_α.

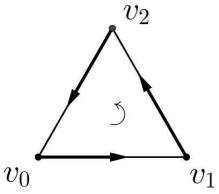
Remark 1.1.11. Traditionally, simplicial homology is defined for **simplicial complexes**, which are the Δ-complexes whose simplices are uniquely determined by their vertices. This results in saying that each n-simplex has $n+1$ distinct vertices, and that no other n-simplex has this same set of vertices. Thus a simplicial complex can be described combinatorially as a set X_0 of vertices together with sets X_n of n-simplices, which are $(n+1)$-element subsets of X_0. The only requirement is that each $(k+1)$-element subset of the vertices of an n-simplex in X_n is a k-simplex, in X_k.
In other words we can say that Δ-complexes are a mild generalization of this more classical notion of a simplicial complex.

Our goal is now to define the simplicial homology groups of a Δ-complex X. As we have seen in the preceding section this means that we need to construct a chain complex, *i.e.* we need to define abelian groups which are connected by a boundary operator.

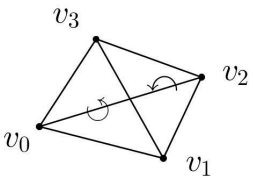
Let $\Delta_n(X)$ be the free abelian group with basis the open n-simplices of X which we denote by e_α^n. Elements of $\Delta_n(X)$, called **n-chains**, can be written as finite formal sums $\sum_\alpha n_\alpha e_\alpha^n$ with coefficients $n_\alpha \in \mathbb{Z}$. Equivalently, we could write $\sum_\alpha n_\alpha \sigma_\alpha$ where $\sigma_\alpha : \Delta_n \to X$ is the characteristic map of e_α^n, with image the closure of e_α^n. Such a sum $\sum_\alpha n_\alpha \sigma_\alpha$ can be thought of as a finite collection, or 'chain', of n-simplices in X with integer multiplicities, the coefficients n_α.
As one can see in the figure below, the boundary of the n-simplex $[v_0, \dots, v_n]$ consists of the various $(n-1)$-dimensional simplices $[v_0, \dots, \hat{v_i}, \dots, v_n]$, where the "hat" symbol over v_i indicates that this vertex is deleted from the sequence $v_0, \dots, v_n$. In terms of chains, we might then wish to say that the boundary of $[v_0, \dots, v_n]$ (*i.e.* here we mean the image of the boundary operator $\delta_n : \Delta_n(X) \to \Delta_{n-1}(X)$) is the $(n-1)$-chain formed by the sum of the faces $[v_0, \dots, \hat{v_i}, \dots, v_n]$. However, in order to fulfil the condition $\delta_n \circ \delta_{n+1} = 0$, for all n, it turns out that we need to insert certain signs and instead let the boundary of $[v_0, \dots, v_n]$ be $\sum_i (-1)^i [v_0, \dots, \hat{v_i}, \dots, v_n]$. Heuristically, the signs are inserted to take orientations into account, so that all the faces of a simplex are coherently oriented, as indicated in the following picture: (Notice for instance that for the 2-simplex the arrow of the edge $[v_0, v_2]$ needs to be inverted, therefore we add a minus sign in front of it.)



$$\delta[v_0, v_1] = [v_1] - [v_0]$$



$$\delta[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$



$$\delta[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$$

With this geometry in mind we define for a general Δ-complex X a **boundary operator** $\delta_n : \Delta_n(X) \to \Delta_{n-1}(X)$ by specifying its values on basis elements:

$$\delta_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha|_{[v_0,\ldots,\hat{v_i},\ldots,v_n]}.$$

Note that the right hand side of this equation does indeed lie in $\Delta_{n-1}(X)$ since each restriction $\sigma_\alpha|_{[v_0,\ldots,\hat{v_i},\ldots,v_n]}$ is the characteristic map of an $(n-1)$-simplex of X.

Example 1.1.12. The composition $\delta_1\delta_2[v_0, v_1, v_2]$ gives 0.
Indeed,

$$\begin{aligned} \delta_1\delta_2[v_0, v_1, v_2] &= \delta_1([v_1, v_2] - [v_0, v_2] + [v_0, v_1]) \\ &= [v_2] - [v_1] - [v_2] + [v_0] + [v_1] - [v_0] \\ &= 0. \end{aligned}$$

The Example before works for any natural number n so that the homomorphism δ really defines a **boundary homomorphism**. In fact we have the following Lemma:

Lemma 1.1.13. *The composition* $\Delta_n(X) \xrightarrow{\delta_n} \Delta_{n-1}(X) \xrightarrow{\delta_{n-1}} \Delta_{n-2}(X)$ *is zero.*

Proof. We have $\delta_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_n]}$, and hence

$$\begin{aligned} \delta_{n-1}\delta_n(\sigma) &= \sum_{j<i} (-1)^i (-1)^j \sigma|_{[v_0,\ldots,\hat{v_j},\ldots,\hat{v_i},\ldots,v_n]} \\ &+ \sum_{j>i} (-1)^i (-1)^{j-1} \sigma|_{[v_0,\ldots,\hat{v_i},\ldots,\hat{v_j},\ldots,v_n]}. \end{aligned}$$

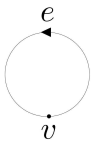
The latter two summations cancel since after switching i and j in the second sum, it becomes the negative of the first. □

All this shows that we just constructed a chain complex $(C_\bullet, \delta_\bullet)$ (where $C_n = \Delta_n(X)$) of free abelian groups generated over a topological space X. Hence, we can consider its n-th homology group $H_n^\Delta(X, \mathbb{Z}) := H_n(C, \mathbb{Z}) = \text{Ker}\, \delta_n / \text{Im}\, \delta_{n+1}$ which is called the **n-th simplicial homology group** of X. Note that we have not got any n-simplex for n being a negative integer. Hence, we have to extend the sequence by 0 at the right end, with $\delta_0 = 0$, *i.e.*

$$\cdots \to \Delta_{n+1}(X) \xrightarrow{\delta_{n+1}} \Delta_n(X) \xrightarrow{\delta_n} \Delta_{n-1}(X) \to \cdots \to \Delta_1(X) \xrightarrow{\delta_1} \Delta_0(X) \xrightarrow{\delta_0} 0.$$

Remark 1.1.14. When we work with free abelian groups generated over a topological space X as it is the case in this section for the chains $\Delta_n(X)$, we will adopt the notation $H_n(X, \mathbb{Z})$ for the **n-th homology group** (resp. $H_*(X) := \bigoplus_n H_n(X, \mathbb{Z})$ for the homology group).

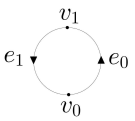
Example 1.1.15. $X = S^1$, with one vertex v and one edge e. Then $\Delta_0(S^1)$ and $\Delta_1(S^1)$ are both $\mathbb{Z}$ and the boundary map δ_1 is zero since $\delta e = v - v$. The groups $\Delta_n(S^1)$ are 0 for $n \geq 2$ since there are no simplices in these dimensions. Hence,



$$H_n^\Delta(S^1, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{for } n = 0, 1 \\ 0 & \text{for } n \geq 2. \end{cases}$$

This is an illustration of the general fact that if the boundary maps in a chain complex are all zero, then the homology groups of the complex are isomorphic to the chain groups themselves.

Example 1.1.16. $X = S^1$, with two vertices v_0, v_1 and two edges e_0, e_1. Then $\Delta_0(S^1)$ and $\Delta_1(S^1)$ are both $\mathbb{Z}^2$ and the groups $\Delta_n(S^1)$ are 0 for $n \geq 2$ since there are no simplices in these dimensions.



Hence, the chain complex $0 \to \Delta_0(S^1) \xrightarrow{\delta_1} \Delta_1(S^1) \to 0$ is isomorphic to the chain complex $0 \to \mathbb{Z}^2 \xrightarrow{\delta_1'} \mathbb{Z}^2 \to 0$ where the identification via the isomorphism is given as follows:

$$v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and we use a similar identification with $\mathbb{Z}^2$ for the edges e_0, e_1. In the chain complex involving tuples of integers, the map δ_1' is a linear map which means

that it can be represented by a matrix. This matrix is entirely defined by the values the map δ_1' takes on the basis vectors. We have:

$$\delta_1(e_0) = v_1 - v_0, \;\; \delta_1(e_1) = v_0 - v_1.$$

Using the identification via the corresponding isomorphism this yields the column vectors

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \; \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

which are the successive column vectors of the matrix A corresponding to the linear map δ_1', *i.e.*

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Now we are ready to compute $H_n^{\Delta}(S^1, \mathbb{Z})$ for $n = 0, 1$. Note that $H_n^{\Delta}(S^1, \mathbb{Z}) = 0$ for $n \geq 2$ for the same reasons as in the Example above. We have $H_0^{\Delta}(S^1, \mathbb{Z}) = \mathbb{Z}^2/\text{Im } \delta_1 = \mathbb{Z}^2/\text{Im } A$. Let us take a look at Im A:

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} -a_0 + a_1 \\ a_0 - a_1 \end{pmatrix} = a_0 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + a_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

But these 2 generating vectors are linearly dependent so that one can conclude that

$$\text{Im } A = \mathbb{Z} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \simeq \mathbb{Z}.$$

If we manage to build a surjective linear map $l : \mathbb{Z}^2 \twoheadrightarrow \mathbb{Z}$ such that Ker $l =$ Im A, the first isomorphism Theorem will allow us to conclude that $\mathbb{Z}^2/\text{Ker } l \simeq$ Im l. Put in another way this exactly yields that, $\mathbb{Z}^2/\text{Im } A \simeq \mathbb{Z}$.

Claim. *The linear map $l : \mathbb{Z}^2 \to \mathbb{Z}$ associated to the 1×2 matrix $L = \begin{pmatrix} 1 & 1 \end{pmatrix}$ has the desired properties.*

Proof. Surjectivity is clear.
Moreover,

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = a_0 + a_1$$

and

$$a_0 + a_1 = 0 \Leftrightarrow a_1 = -a_0,$$

so that

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} a_0 \\ -a_0 \end{pmatrix} = a_0 \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

which finally shows that

$$\text{Ker } l = \left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle_{\mathbb{Z}} = \text{Im } A.$$

□

Finally, we just showed that

$$\boxed{H_0^\Delta(S^1,\mathbb{Z}) = \mathbb{Z}^2/\text{Im } A \simeq \mathbb{Z}.}$$

On the other hand we have that $H_1^\Delta(S^1,\mathbb{Z}) \simeq \text{Ker } \delta_1 = \text{Ker } A$. Let us compute Ker A:

$$\begin{aligned} A\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} &\Leftrightarrow \begin{pmatrix} -a_0 + a_1 \\ a_0 - a_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &\Leftrightarrow \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \mathbb{Z}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

We have just shown that Ker $A \simeq \mathbb{Z}$, so finally

$$\boxed{H_1^\Delta(S^1,\mathbb{Z}) \simeq \text{Ker } A \simeq \mathbb{Z}.}$$

Altogether we just got the same conclusions as in the Example before, namely

$$H_n^\Delta(S^1,\mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{for } n = 0,1 \\ 0 & \text{for } n \geq 2. \end{cases}$$

Example 1.1.17. Let us have a closer look at $H_0^\Delta(X,\mathbb{Z})$ for an arbitrary topological space X. By definition $H_0^\Delta(X,\mathbb{Z}) = \text{Ker } \delta_0/\text{Im } \delta_1 = \Delta_0(X)/\text{Im } \delta_1$, where

$$\Delta_0(X) = \left\{ \sum_\alpha n_\alpha e_\alpha^0 \mid n_\alpha \in \mathbb{Z} \text{ and } e_\alpha^0 \text{ is an open } 0-\text{simplex of } X \right\}.$$

Hence, $\sum_\alpha n_\alpha e_\alpha^0$, $\sum_\alpha m_\alpha e_\alpha^0 \in \Delta_0(X)$ belong to the same class if and only if they differ by an element in Im δ_1, *i.e.*

$$\sum_\alpha (n_\alpha - m_\alpha) e_\alpha^0 = \delta_1 \left(\sum_\beta n_\beta e_\beta^1 \right).$$

However, for any open 1-simplex joining the points p and q we have that $\delta_1([p,q]) = q - p$ so that $\bar{p} = \bar{q}$ in $H_0^\Delta(X,\mathbb{Z})$. From this it follows that if our topological space X is path connected, we have an isomorphism $H_0^\Delta(X,\mathbb{Z}) \simeq \mathbb{Z}$. However if our topological space X is not path connected we have that $H_0^\Delta(X,\mathbb{Z}) \simeq \mathbb{Z}^n$ where n equals the number of path connected components.

1.2 Cohomology

Cohomology is an algebraic variant of homology, the result of a simple dualization in the definition. This dualization is done by the left exact contravariant Hom functor Hom$(-,\mathbb{Z})$. (cf. A.14)

Since we work here with a contravariant functor the induced homomorphisms will go in the opposite direction. To be more precise consider a completely general chain complex $(C_\bullet, \delta_\bullet)$ of abelian groups: (This could be for instance the chain complex related to simplicial homology.)

$$\cdots \to C_{n+1} \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \to \cdots .$$

To dualize this complex we replace each chain group C_n by its dual **cochain group** $C_n^* := \mathrm{Hom}(C_n, \mathbb{Z})$, the group of homomorphisms $C_n \to \mathbb{Z}$, and we replace each boundary map $\delta_n : C_n \to C_{n-1}$ by its dual **coboundary map** $d^{n-1} := \delta_n^* : C_{n-1}^* \to C_n^*$. The reason why d goes in the opposite direction from δ, increasing rather than decreasing dimensions, is purely formal: For a homomorphism $\alpha : A \to B$, the dual homomorphism $\alpha^* : \mathrm{Hom}(B, \mathbb{Z}) \to \mathrm{Hom}(A, \mathbb{Z})$ is defined by $\alpha^*(\varphi) = \varphi\alpha$, so α^* sends $B \xrightarrow{\varphi} \mathbb{Z}$ to the composition $A \xrightarrow{\alpha} B \xrightarrow{\varphi} \mathbb{Z}$. Dual homomorphisms obviously satisfy $(\alpha\beta)^* = \beta^*\alpha^*$, $\mathbb{1}^* = \mathbb{1}$, and $0^* = 0$. In particular, since $\delta\delta = 0$ it follows that $dd = 0$ and we end up with what we call a **cochain complex** which allows us to define the **cohomology group**.

Definition 1.2.1 (**Cochain complex**)**.** A sequence

$$\cdots \leftarrow C_{n+1}^* \xleftarrow{d^n} C_n^* \xleftarrow{d^{n-1}} C_{n-1}^* \leftarrow \cdots$$

of free abelian groups C_n^*, $n \in \mathbb{Z}$, connected by homomorphisms
$d^n : C_n^* \to C_{n+1}^*$ such that the composition of any two consecutive maps is zero (*i.e.* $d^n \circ d^{n-1} = 0$ for all n) is called a **cochain complex**. We will denote such a sequence by $(C_\bullet^*, d^\bullet)$.

Definition 1.2.2. The elements of C_n^* are the **cochains** (resp. the cochains of degree n), the elements of $\mathrm{Ker}\, d^n \subseteq C_n^*$ are the **cocycles**, and those of $\mathrm{Im}\, \delta_{n-1} \subseteq C_n^*$ are referred to as **coboundaries**. The operator d is said to be a **coboundary operator** or a **cohomology operator**.

Remark 1.2.3. The map $d^n : C_n^* \to C_{n+1}^*$ increases the cochain degree by 1. We then say that d has degree $+1$.
The condition $d^n \circ d^{n-1} = 0$ for all n, means exactly that $\mathrm{Im}\, d^{n-1} \subseteq \mathrm{Ker}\, d^n$, so that the quotient space $\mathrm{Ker}\, d^n / \mathrm{Im}\, d^{n-1}$ makes sense.

Definition 1.2.4. We call the quotient group

$$H^n(C, \mathbb{Z}) := {}^{\mathrm{Ker}\, d^n}\!\!\diagup\!\!_{\mathrm{Im}\, d^{n-1}}$$

the n**-th cohomology group** of the cochain complex $(C_\bullet^*, d^\bullet)$. The direct sum

$$H^*(X) := \bigoplus_n H^n(C, \mathbb{Z})$$

is referred to as the **cohomology group** of $(C_\bullet^*, d^\bullet)$.

Remark 1.2.5. Notice that for cohomology we use superscript notation contrary to homology where we use subscript notation.

We have just seen how we can derive the cochain complex from a given chain complex in order to compute the related cohomology groups. The next Theorem shows that the cohomology groups $H^n(C,\mathbb{Z})$ are even determined exclusively by $\mathbb{Z}$ and the homology groups $H_n(C,\mathbb{Z})$. This means that we can extract the cohomology groups right from computations with homology groups. This result is known as the **universal coefficient Theorem for cohomology**.

Theorem 1.2.6. *If a chain complex $(C_\bullet, \delta_\bullet)$ of free abelian groups has homology groups $H_n(C,\mathbb{Z})$, then the cohomology groups $H^n(C,\mathbb{Z})$ of the cochain complex* $\mathrm{Hom}(C_n,\mathbb{Z})$ *are determined by split exact sequences*

$$0 \to \mathrm{Ext}^1(H_{n-1}(C,\mathbb{Z}),\mathbb{Z}) \to H^n(C,\mathbb{Z}) \xrightarrow{h} \mathrm{Hom}(H_n(C,\mathbb{Z}),\mathbb{Z}) \to 0.$$

The reader who is interested in the proof of this Theorem can have a look at [7] p.190-195.

Remark 1.2.7. The preceding Theorem involves the knowledge of the **extension functors** which are derived functors of Hom functors. The interested reader may have a look at [18] in order to catch up with the notion of derived functors.

Remark 1.2.8. Theorem 1.2.6 is of particular interest when the extension functor vanishes.

$$0 \to \mathrm{Ext}^1(H_{n-1}(C,\mathbb{Z}),\mathbb{Z}) \xrightarrow{\alpha} H^n(C,\mathbb{Z}) \xrightarrow{h} \mathrm{Hom}(H_n(C,\mathbb{Z}),\mathbb{Z}) \to 0$$

is an exact sequence hence, $\mathrm{Im}\,\alpha = \mathrm{Ker}\,h$. Since $\mathrm{Ext}^1(H_{n-1}(C,\mathbb{Z}),\mathbb{Z}) = \{0\}$, we have that $\mathrm{Im}\,\alpha = \{0\}$, in particular $\mathrm{Ker}\,h = \{0\}$. Now, applying the first isomorphism Theorem for the homomorphism h yields the isomorphism

$$H^n(C,\mathbb{Z})\big/_{\mathrm{Ker}\,(h)} \simeq \mathrm{Hom}(H_n(C,\mathbb{Z}),\mathbb{Z}).$$

Finally, since $\mathrm{Ker}\,h = \{0\}$ we get that

$$H^n(C,\mathbb{Z}) \simeq \mathrm{Hom}(H_n(C,\mathbb{Z}),\mathbb{Z}).$$

In the following subsection we will apply the theory we have developed here to the case of simplicial homology in order to define the **simplicial cohomology**.

1.2.1 Simplicial Cohomology

Let X be a topological space equipped with a Δ-complex structure. We define the group $\Delta^n(X)$ of **simplicial n-cochains with coefficients in $\mathbb{Z}$** to be the dual group $\mathrm{Hom}(\Delta_n(X), \mathbb{Z})$ of the simplicial chain group $\Delta_n(X)$. Thus an n-cochain $\varphi \in \Delta^n(X)$ assigns to each simplicial n-simplex $\sigma : \Delta_n \to X$ a value $\varphi(\sigma) \in \mathbb{Z}$. Since the simplicial n-simplices form a basis for $\Delta_n(X)$, these values can be chosen arbitrarily, hence n-cochains are exactly equivalent to functions from simplicial n-simplices to $\mathbb{Z}$.
The **coboundary map** $d^n : \Delta^n(X) \to \Delta^{n+1}(X)$ is the dual δ^*_{n+1}, so for a cochain $\varphi \in \Delta^n(X)$, its coboundary $d\varphi$ is the composition $\Delta_{n+1}(X) \xrightarrow{\delta_{n+1}} \Delta_n(X) \xrightarrow{\varphi} \mathbb{Z}$. This means that for a simplicial $(n+1)$-simplex $\sigma : \Delta_{n+1} \to X$ we have:

$$d\varphi(\sigma) = \sum_i (-1)^i \varphi\left(\sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{n+1}]}\right).$$

It is automatic that $d^2 = 0$ since d^2 is the dual of $\delta^2 = 0$.
All this shows that we just constructed a cochain complex $(C^*_\bullet, d^\bullet)$ (where $C^*_n = \Delta^n(X)$) of free abelian groups generated over a topological space X. Hence, we can consider its n-th cohomology group $H^n_\Delta(X, \mathbb{Z}) := H^n(C, \mathbb{Z}) = \mathrm{Ker}\, d^n / \mathrm{Im}\, d^{n-1}$ which is called the **n-th simplicial cohomology group** of X.

$$\cdots \leftarrow \Delta^{n+1}(X) \xleftarrow{d^n} \Delta^n(X) \xleftarrow{d^{n-1}} \Delta^{n-1}(X) \leftarrow \cdots \leftarrow \Delta^1(X) \xleftarrow{d^0} \Delta^0(X) \leftarrow 0$$

For a cochain φ to be a cocycle means that $d\varphi = \varphi\delta = 0$, or in other words, φ vanishes on boundaries.

Example 1.2.9 (Application of the universal coefficient Theorem for cohomology)**.** Since the chain groups $\Delta_n(X)$ are free, the universal coefficient Theorem guarantees the existence of split short exact sequences

$$0 \to \mathrm{Ext}^1(H^\Delta_{n-1}(X, \mathbb{Z}), \mathbb{Z}) \to H^n_\Delta(X, \mathbb{Z}) \xrightarrow{h} \mathrm{Hom}(H^\Delta_n(X, \mathbb{Z}), \mathbb{Z}) \to 0.$$

When $n = 0$ there is no Ext term, and as we have established in Remark 1.2.8 the universal coefficient Theorem reduces to an isomorphism $H^0_\Delta(X, \mathbb{Z}) \simeq \mathrm{Hom}(H^\Delta_0(X, \mathbb{Z}), \mathbb{Z})$. In Example 1.1.17 we have discovered that $H^\Delta_0(X, \mathbb{Z}) \simeq \mathbb{Z}^n$ where n is the number of connected components of X. Moreover, we have that $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}) \simeq \mathrm{Hom}(\bigoplus_i^n \mathbb{Z}, \mathbb{Z}) \simeq \bigoplus_i^n \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \simeq \bigoplus_i^n \mathbb{Z} \simeq \mathbb{Z}^n$. Finally, this means that $H^0_\Delta(X, \mathbb{Z}) \simeq \mathrm{Hom}(H^\Delta_0(X, \mathbb{Z}), \mathbb{Z}) \simeq \mathbb{Z}^n$. Similarly in the case of $H^1_\Delta(X, \mathbb{Z})$ the universal coefficient Theorem gives an isomorphism $H^1_\Delta(X, \mathbb{Z}) \simeq \mathrm{Hom}(H^\Delta_1(X, \mathbb{Z}), \mathbb{Z})$ since $\mathrm{Ext}^1(H^\Delta_0(X, \mathbb{Z}), \mathbb{Z}) = 0$.

In the case where X is in particular a complex manifold of dimension n we have the following interesting property:

Proposition 1.2.10. *Let X be a complex manifold of dimension n, i.e. $dim_{\mathbb{C}}X = n$. Then,*

$$H^i_\Delta(X, \mathbb{Z}) = 0 \quad \forall i > 2n.$$

Remark 1.2.11. The same assertion holds true in the homological setting, *i.e.* for a complex manifold X of dimension n we have that

$$H_i^\Delta(X, \mathbb{Z}) = 0 \quad \forall i > 2n.$$

The interested reader can find a proof of this result in [7] Theorem 3.26. p.236.

The proof of Proposition 1.2.10 can be deduced from the homological case and is an application of the universal coefficient Theorem 1.2.6. Since X is a complex manifold, we know in particular that it is orientable. Corollary 3.28. of [7] tells us then that $H^\Delta_{n-1}(X, \mathbb{Z})$ is torsion free which allows us to conclude that it is a free abelian group and so we have that $H^\Delta_{n-1}(X, \mathbb{Z}) \simeq \mathbb{Z}^k$. This again tells us that $\mathrm{Ext}^1(H^\Delta_{n-1}(X, \mathbb{Z}), \mathbb{Z}) = 0$. Hence, as we have already seen in Remark 1.2.8 we have that $H^n_\Delta(X, \mathbb{Z}) \simeq \mathrm{Hom}(H^\Delta_n(X, \mathbb{Z}), \mathbb{Z})$. However, we already know that $H^\Delta_n(X, \mathbb{Z}) = 0 \quad \forall i > 2n$. Finally, this means that we have just shown that $H^n_\Delta(X, \mathbb{Z}) = 0 \quad \forall i > 2n$.

We will close this section by some important considerations around homology and cohomology theories. Right now, we have introduced simplicial homology which is an example of a homology theory, respectively simplicial cohomology which is an example of a cohomology theory. It turns out that the simplicial homology satisfies some axioms which are called the *Eilenberg-Steenrod* axioms.

Eilenberg-Steenrod axioms. Let $\mathcal{C}$ be the category of pairs of topological spaces. A **homology theory** $(H, \mathcal{C})$ on $\mathcal{C}$ is a sequence of functors $H_n : \mathcal{C} \to \mathbf{Ab}$ for $n \geq 0$ and a sequence of natural transformations $\delta_n : H_n \to H_{n-1} \circ R$ (where $R : \mathcal{C} \to \mathcal{C}$ is the functor $(X, A) \mapsto (A, \emptyset)$) such that the following axioms hold.

- **Homotopy Axiom.** If $f_0, f_1 : (X, A) \to (Y, B)$ are homotopic (i.e., there is a homotopy $F : (X \times \mathbf{I}, A \times \mathbf{I}) \to (Y, B)$ with $F_0 = f_0$ and $F_1 = f_1$), then $H_n(f_0) = H_n(f_1) : H_n(X, A) \to H_n(Y, B)$ for all $n \geq 0$.
- **Exactness Axiom.** For every pair (X, A) with inclusions $i : (A, \emptyset) \hookrightarrow (X, \emptyset)$ and $j : (X, \emptyset) \hookrightarrow (X, A)$, there is an exact sequence

$$\cdots \to H_n(A, \emptyset) \xrightarrow{H_n(i)} H_n(X, \emptyset) \xrightarrow{H_n(j)} H_n(X, A) \xrightarrow{\delta_n} H_{n-1}(A, \emptyset) \to \cdots .$$

- **Excision Axiom.** For every pair (X, A) and every open subset U of X with $\bar{U} \subset \mathring{A}$, the inclusion $(X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces isomorphisms
$$H_n(X \setminus U, A \setminus U) \xrightarrow{\simeq} H_n(X, A)$$
for all $n \geq 0$.

- **Dimension Axiom.** If X is a one point space, then $H_n(X, \emptyset) = 0$ for all $n > 0$. One calls $H_0(X, \emptyset)$ the **coefficient group**. In our case we have that $H_0(X, \emptyset) = \mathbb{Z}$ (have a look at Example 1.1.15 and Example 1.1.16.) so that we agree to denote the coefficient group by $H_0(X, \mathbb{Z})$.

If one fixes the coefficient group, any two different homology theories satisfying these specific axioms will actually have isomorphic homology groups. For the rest of the paper we will denote by $H_n(X, \mathbb{Z})$ the homology group of some homology theory which satisfies the *Eilenberg-Steenrod* axioms with coefficient group $\mathbb{Z}$. In particular we then have that $H_n^{\Delta}(X, \mathbb{Z}) \simeq H_n(X, \mathbb{Z})$. By the dualization property of cohomology it is then immediate that we also that $H^n_{\Delta}(X, \mathbb{Z}) \simeq H^n(X, \mathbb{Z})$. The interested reader who wants to have more details about the *Eilenberg-Steenrod* axioms can have a look at [15] p.231.

1.3 Cup Product

We are now ready to introduce the **cup product** which provides the multiplication in $H^*(X)$ and turns it into a so-called graded-commutative ring. It is important to mention that this product is only defined in the setting of cohomology so that it has no application for homology. In what follows we will work with the simplicial cohomology we introduced in section 1.2.1. For the understanding of the following definition the theory developed in that section may be useful.

Definition 1.3.1. For cochains $\varphi \in \Delta^k(X)$ and $\psi \in \Delta^l(X)$, the **cup product** $\varphi \smile \psi \in \Delta^{k+l}(X)$ is the cochain whose value on a simplicial simplex $\sigma : \Delta_{k+l} \to X$ is given by the formula

$$(\varphi \smile \psi)(\sigma) = \varphi\left(\sigma|_{[v_0,\ldots,v_k]}\right) \psi\left(\sigma|_{[v_k,\ldots,v_{k+l}]}\right),$$

where the right-hand side is the product in $\mathbb{Z}$. We then extend this definition by bilinearity.

To see that this cup product of cochains induces a cup product of cohomology classes we need the following formula relating it to the coboundary map:

Lemma 1.3.2. *$d(\varphi \smile \psi) = d\varphi \smile \psi + (-1)^k \varphi \smile d\psi$ for $\varphi \in \Delta^k(X)$ and $\psi \in \Delta^l(X)$.*

Proof. (Based on Lemma 3.6. in [7].) For $\sigma : \Delta_{k+l+1} \to X$ we have:

$$(d\varphi \smile \psi)(\sigma) = \sum_{i=0}^{k+1} (-1)^i \varphi\left(\sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{k+1}]}\right) \psi\left(\sigma|_{[v_{k+1},\ldots,v_{k+l+1}]}\right),$$

$$(-1)^k(\varphi \smile d\psi)(\sigma) = \sum_{i=k}^{k+l+1} (-1)^i \varphi\left(\sigma|_{[v_0,\ldots,v_k]}\right) \psi\left(\sigma|_{[v_k,\ldots,\hat{v_i},\ldots,v_{k+l+1}]}\right).$$

When we add these two expressions, the last term of the first sum cancels out the first term of the second sum, and the remaining terms are exactly $d(\varphi \smile \psi)(\sigma) = (\varphi \smile \psi)(\delta\sigma)$ since $\delta\sigma = \sum_{i=0}^{k+l+1}(-1)^i\sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{k+l+1}]}$. □

Exercise 1.3.3. Let us check in detail the last affirmation of the preceding proof:

$$\begin{aligned}
&(\varphi \smile \psi)(\delta\sigma)\\
&= (\varphi \smile \psi)\left(\sum_{i=0}^{k+l+1} (-1)^i \sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{k+l+1}]}\right)\\
&= (\varphi \smile \psi)\left(\sum_{i=0}^{k} (-1)^i \sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{k+l+1}]} + \sum_{i=k+1}^{k+l+1} (-1)^i \sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{k+l+1}]}\right)\\
&= (\varphi \smile \psi)\left(\sum_{i=0}^{k} (-1)^i \sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{k+l+1}]}\right)\\
&+ (\varphi \smile \psi)\left(\sum_{i=k+1}^{k+l+1} (-1)^i \sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{k+l+1}]}\right)\\
&= \sum_{i=0}^{k} (-1)^i (\varphi \smile \psi)\left(\sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{k+l+1}]}\right)\\
&+ \sum_{i=k+1}^{k+l+1} (-1)^i (\varphi \smile \psi)\left(\sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{k+l+1}]}\right)\\
&= \sum_{i=0}^{k} (-1)^i \varphi\left(\sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{k+1}]}\right) \psi\left(\sigma|_{[v_{k+1},\ldots,v_{k+l+1}]}\right)\\
&+ \sum_{i=k+1}^{k+l+1} (-1)^i \varphi\left(\sigma|_{[v_0,\ldots,v_k]}\right) \psi\left(\sigma|_{[v_k,\ldots,\hat{v_i},\ldots,v_{k+l+1}]}\right).
\end{aligned}$$

Note that we used here nothing else than the property that $\varphi \smile \psi$ is a cochain, *i.e.* a homomorphism.

From the formula $d(\varphi \smile \psi) = d\varphi \smile \psi \pm \varphi \smile d\psi$ of the preceding Lemma it is apparent that the cup product of two cocycles is again a cocycle. Also, the cup product of a cocycle and a coboundary, in either order, is a coboundary since $\varphi \smile d\psi = \pm d(\varphi \smile \psi)$ if $d\varphi = 0$ and $d\varphi \smile \psi = d(\varphi \smile \psi)$ if $d\psi = 0$. It follows that there is an induced cup product

$$H^k(X, \mathbb{Z}) \times H^l(X, \mathbb{Z}) \xrightarrow{\smile} H^{k+l}(X, \mathbb{Z}).$$

This is associative and distributive since at the level of cochains the cup product obviously has these properties. Since $\mathbb{Z}$ has an identity element, there is an identity element for the cup product, the class $\mathbb{1} \in H^0(X, \mathbb{Z})$ defined to be the 0-cocycle taking the value 1 on each simplicial 0-simplex. All of this shows that the cup product fulfils all the needed properties in order to turn the cohomology group $H^*(X) := \bigoplus_n H^n(X, \mathbb{Z})$ into a **graded ring**. (The grading property is clear and is given by the decomposition into the direct sum.) Elements of $H^*(X)$ are finite sums $\sum_i \alpha_i$ with $\alpha_i \in H^i(X, \mathbb{Z})$, and the product of two such sums is defined to be $\left(\sum_i \alpha_i\right)\left(\sum_j \beta_j\right) = \sum_{i,j} \alpha_i \beta_j$.

Exercise 1.3.4. Let us check that the cup product is indeed associative and distributive.
Let $\varphi \in \Delta^k(X), \psi \in \Delta^l(X), \xi \in \Delta^m(X)$ and $\sigma \in \Delta_{k+l+m}$.

- Associativity:

$$\begin{aligned}((\varphi \smile \psi) \smile \xi)(\sigma) &= (\varphi \smile \psi)\left(\sigma|_{[v_0,\ldots,v_{k+l}]}\right) \psi\left(\sigma|_{[v_{k+l},\ldots,v_{k+l+m}]}\right) \\ &= \varphi\left(\sigma|_{[v_0,\ldots,v_k]}\right) \psi\left(\sigma|_{[v_k,\ldots,v_{k+l}]}\right) \xi\left(\sigma|_{[v_{k+l},\ldots,v_{k+l+m}]}\right),\end{aligned}$$

$$\begin{aligned}(\varphi \smile (\psi \smile \xi))(\sigma) &= \varphi\left(\sigma|_{[v_0,\ldots,v_k]}\right) (\psi \smile \xi)\left(\sigma|_{[v_k,\ldots,v_{k+l+m}]}\right) \\ &= \varphi\left(\sigma|_{[v_0,\ldots,v_k]}\right) \psi\left(\sigma|_{[v_k,\ldots,v_{k+l}]}\right) \xi\left(\sigma|_{[v_{k+l},\ldots,v_{k+l+m}]}\right).\end{aligned}$$

- Distributivity:
  Here we need to distinguish 2 cases:

  1. if $k \neq l$:
     This follows immediately from Definition 1.3.1 where we indicated that we extend the cup product by bilinearity. This exactly means that $(\varphi + \psi) \smile \xi = \varphi \smile \xi + \psi \smile \xi$ and that $\varphi \smile (\psi + \xi) = \varphi \smile \psi + \varphi \smile \xi$.

2. <u>if $k = l$</u>

$$\begin{aligned}
&((\varphi + \psi) \smile \xi)(\sigma) \\
&= (\varphi + \psi)\left(\sigma|_{[v_0,\dots,v_k]}\right) \xi \left(\sigma|_{[v_k,\dots,v_{k+m}]}\right) \\
&= \left(\varphi\left(\sigma|_{[v_0,\dots,v_k]}\right) + \psi\left(\sigma|_{[v_0,\dots,v_k]}\right)\right) \xi \left(\sigma|_{[v_k,\dots,v_{k+m}]}\right) \\
&= \varphi\left(\sigma|_{[v_0,\dots,v_k]}\right) \xi \left(\sigma|_{[v_k,\dots,v_{k+m}]}\right) \\
&+ \psi\left(\sigma|_{[v_0,\dots,v_k]}\right) \xi \left(\sigma|_{[v_k,\dots,v_{k+m}]}\right) \\
&= (\varphi \smile \xi)(\sigma) + (\psi \smile \xi)(\sigma).
\end{aligned}$$

And a similar computation gives linearity in the other argument.

Exercise 1.3.5. Let us check in more details (in a computational way) that the cup product defined on cochains induces a cup product at the level of cohomology groups.
Definition 1.3.1 yields a map

$$\Delta^k(X) \times \Delta^l(X) \xrightarrow{\smile} \Delta^{k+l}(X).$$

We want to construct a map

$$H^k_\Delta(X, \mathbb{Z}) \times H^l_\Delta(X, \mathbb{Z}) \xrightarrow{\smile} H^{k+l}_\Delta(X, \mathbb{Z}).$$

Recall that $H^n_\Delta(X, \mathbb{Z}) = \text{Ker } d^n / \text{Im } d^{n-1}$, so we first need to check the following:

$$d^k a = 0, \;\; d^l b = 0 \Rightarrow d^{k+l}(a \smile b) = 0.$$

This follows from the formula of Lemma 1.3.2 which tells us that the cup product of two cocycles is again a cocycle. All of this allows us now to take classes; in other words we can factor out the coboundaries. Now it remains to show that this is independent of the choice of the representative of a given class. Therefore let $\left(\bar{a}, \bar{b}\right) = \left(\bar{a'}, \bar{b'}\right) \in H^k_\Delta(X, \mathbb{Z}) \times H^l_\Delta(X, \mathbb{Z})$, *i.e.* $a - a' = d\eta$ and $b - b' = d\xi$. We want to show that $a \smile b - a' \smile b' = d\nu$. We have:

$$\begin{aligned}
a \smile b - a' \smile b' &= (a' + d\eta) \smile (b' + d\xi) - a' \smile b' \\
&= a' \smile b' + a' \smile d\xi + d\eta \smile b' + d\eta \smile d\xi - a' \smile b' \\
&= a' \smile d\xi + d\eta \smile b' + d\eta \smile d\xi.
\end{aligned}$$

Now we use the property that the cup product of a cocycle and a coboundary, in either order, is a coboundary (this follows from Lemma 1.3.2). Note that a coboundary is also a cocycle since $\text{Im } d^n \subseteq \text{Ker } d^{n+1}$. Hence, the last term

of the preceding sum can also be seen as the cup product of a cocycle and a coboundary. Altogether we get:

$$\begin{aligned} a \smile b - a' \smile b' &= a' \smile d\xi + d\eta \smile b' + d\eta \smile d\xi \\ &= d\nu_1 + d\nu_2 + d\nu_3 \\ &= d\nu, \end{aligned}$$

where in the last equality we just used the fact that d is a homomorphism of abelian groups, *i.e.* we have that $\nu = \nu_1 + \nu_2 + \nu_3$.

The natural question of whether the cup product is commutative is answered by the following Theorem:

Theorem 1.3.6. *For all $\alpha \in H^k(X, \mathbb{Z})$ and all $\beta \in H^l(X, \mathbb{Z})$ we have the following identity:*

$$\alpha \smile \beta = (-1)^{kl} \beta \smile \alpha.$$

The interested reader can find a proof of this assertion in [7] Theorem 3.11. p.210-212.

A graded ring satisfying the commutativity property of the preceding Theorem is usually called **simply commutative** in the context of algebraic topology, in spite of the potential for misunderstanding. In this paper we will prefer to use the terms **graded commutative** or **skew commutative**.

Conclusion: $H^*(X) := \bigoplus_n H^n(X, \mathbb{Z})$ is a **graded skew commutative graded ring**.

1.4 Application: The cohomology ring $H^*(\mathbb{CP}^n)$ of the n-dimensional complex projective space

In this section we want to compute the cohomology ring of the complex projective space of dimension n. To begin with we will take an interest into $H^*(\mathbb{CP}^1)$ where we are going to do the computations explicitly. Afterwards we will state the general result for arbitrary dimensions which is obtained by induction on the dimension n.

1.4.1 The cohomology ring $H^*(\mathbb{CP}^1)$

As we have developed in our theoretical part we will first take an insight into the homology groups before computing the cohomology groups via our dualization process. Then we are able to compute the cup product in order

to characterize our cohomology ring $H^*(\mathbb{CP}^1)$. In what follows we will work with the simplicial homology and simplicial cohomology which we studied in 1.1.1 respectively 1.2.1.
The one dimensional complex projective space $\mathbb{CP}^1$ is isomorphic to the Riemann sphere $\hat{\mathbb{C}}$ which again is isomorphic to S^2. Finally, S^2 is isomorphic to the boundary of the standard 3-simplex so that a Δ-complex structure is given by means of

(i) Four 0-simplices: $E_1^0 := [v_0]$, $E_2^0 := [v_1]$, $E_3^0 := [v_2]$, $E_4^0 := [v_3]$,

(ii) Six 1-simplices: $E_1^1 := [v_0, v_1]$, $E_2^1 := [v_0, v_2]$, $E_3^1 := [v_0, v_3]$, $E_4^1 := [v_1, v_2]$, $E_5^1 := [v_1, v_3]$, $E_6^1 := [v_2, v_3]$,

(iii) Four 2-simplices: $E_1^2 := [v_1, v_2, v_3]$, $E_2^2 := [v_0, v_2, v_3]$, $E_3^2 := [v_0, v_1, v_3]$, $E_4^2 := [v_0, v_1, v_2]$.

According to our notation let us indicate by $\Delta_n(\mathbb{CP}^1)$ the free abelian group with basis the open n-simplices of $\mathbb{CP}^1$ which we denote as usual by $e_\alpha^n := \mathring{E}_\alpha^n$:

$$\Delta_0(\mathbb{CP}^1) = \langle e_1^0, e_2^0, e_3^0, e_4^0 \rangle_{\mathbb{Z}} \simeq \mathbb{Z}^4,$$
$$\Delta_1(\mathbb{CP}^1) = \langle e_1^1, e_2^1, e_3^1, e_4^1, e_5^1, e_6^1 \rangle_{\mathbb{Z}} \simeq \mathbb{Z}^6,$$
$$\Delta_2(\mathbb{CP}^1) = \langle e_1^2, e_2^2, e_3^2, e_4^2 \rangle_{\mathbb{Z}} \simeq \mathbb{Z}^4.$$

Hence, the chain complex

$$0 \to \Delta_2(\mathbb{CP}^1) \xrightarrow{\delta_2} \Delta_1(\mathbb{CP}^1) \xrightarrow{\delta_1} \Delta_0(\mathbb{CP}^1) \xrightarrow{\delta_0} 0$$

is isomorphic to the chain complex

$$0 \to C_2 := \mathbb{Z}^4 \xrightarrow{\delta_2'} C_1 := \mathbb{Z}^6 \xrightarrow{\delta_1'} C_0 := \mathbb{Z}^4 \xrightarrow{\delta_0'} 0.$$

From this it follows that

$$H_i^\Delta(\mathbb{CP}^1, \mathbb{Z}) \simeq H_i(C, \mathbb{Z}) \text{ for } i = 0, 1, 2$$

i.e. the induced homology groups of both complexes are also isomorphic to each other so that it is sufficient to study the complex build up by tuples of integers. In this chain complex $(C_\bullet, \delta_\bullet')$ the maps $\delta_\bullet'$ are just linear maps and we know that they are entirely defined once we know the values on the basis vectors. Also linear maps $l : \mathbb{Z}^n \to \mathbb{Z}^m$ can just be represented by $m \times n$ matrices whose i^{th} column is exactly the image of the corresponding i^{th} basis vector. (cf. 1.1.16)

By identification via the isomorphism between the chain groups we will get that the basis of C_2 is given by the following correspondence:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \simeq e_1^2, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \simeq e_2^2, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \simeq e_3^2, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \simeq e_4^2.$$

In the same way we also obtain the basis of C_1 respectively C_0:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \simeq e_1^1, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \simeq e_2^1, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \simeq e_3^1, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \simeq e_4^1, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \simeq e_5^1, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \simeq e_6^1,$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \simeq e_1^0, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \simeq e_2^0, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \simeq e_3^0, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \simeq e_4^0.$$

Recall that the boundary operator $\delta_n : \Delta_n(\mathbb{CP}^1) \to \Delta_{n-1}(\mathbb{CP}^1)$ for this Δ-complex structure on $\mathbb{CP}^1$ is given by the rule:

$$\delta_n(\sigma_\alpha) = \sum_i (-1)^i \, \sigma_\alpha|_{[v_0,\ldots,\hat{v_i},\ldots,v_n]}.$$

We will now see how to compute with all these tools in order to get the different columns of the involved matrices. Here it is important to keep in mind the geometry we introduced at page 6; to give an example we have that

$$\delta_1[v_0, v_1] = [v_1] - [v_0],$$

so that

$$\delta_1(e_1^1) = e_2^0 - e_1^0.$$

By identification via the corresponding isomorphism we conclude that

$$\delta_1' \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

which gives us the first column vector of the 4×6 matrix corresponding to the linear map δ_1'.

Applying this procedure to the different basis vectors of C_1 we conclude that the 4×6 matrix A corresponding to the linear map δ_1' has the following form:

$$A = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Again, applying this procedure now to the different basis vectors of C_2 and the linear map δ_2' we conclude that the 6×4 matrix B corresponding to the linear map δ_2' has the following form:

$$B = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

We are now ready to understand how to compute the different homology groups of the complex $(C_\bullet, \delta_\bullet')$:

(i) $\underline{H_2(C,\mathbb{Z})}$:

$H_2(C,\mathbb{Z}) = \text{Ker } \delta_2'/\text{Im } \delta_3' \simeq \text{Ker } \delta_2'$ since $\text{Im } \delta_3' = 0$.

So it suffices to compute the kernel of the map δ_2'; but this is the same as computing the kernel of the matrix B. Applying Gauss elimination to the matrix B yields the following reduced row echelon form of it:

$$B' = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$B' \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} a_0 + a_3 \\ a_1 - a_3 \\ a_2 + a_3 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -a_3 \\ a_3 \\ -a_3 \\ a_3 \end{pmatrix},$$

so that

$$\text{Ker } \delta_2' = \mathbb{Z} \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix},$$

i.e. $\text{Ker } \delta_2' \simeq \mathbb{Z}$, so that finally

$$\boxed{H_2^{\Delta}(\mathbb{CP}^1, \mathbb{Z}) \simeq H_2(C, \mathbb{Z}) \simeq \text{Ker } \delta_2' \simeq \mathbb{Z}.}$$

.

(ii) $\underline{H_0(C, \mathbb{Z})}$:

$H_0(C, \mathbb{Z}) = \text{Ker } \delta_0'/\text{Im } \delta_1' \simeq \mathbb{Z}^4/\text{Im } A$.
In order to compute $\text{Im } A$ we this time take a look at the reduced column echelon form of A which is given by:

$$A' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 \end{pmatrix},$$

so that

$$A' \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ -a_0 - a_1 - a_2 \end{pmatrix}$$

$$= a_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + a_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix},$$

which finally shows that

$$\text{Im } A = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\rangle_{\mathbb{Z}}.$$

If we manage to build a surjective linear map $l : \mathbb{Z}^4 \twoheadrightarrow \mathbb{Z}$ such that $\text{Ker } l = \text{Im } A$, the first isomorphism Theorem will allow us to conclude that $\mathbb{Z}^4/\text{Ker } l \simeq \text{Im } l$. Put in another way, this exactly yields that $\mathbb{Z}^4/\text{Im } A \simeq \mathbb{Z}$.

Claim. *The linear map* $l : \mathbb{Z}^4 \to \mathbb{Z}$ *associated to the* 1×4 *matrix* $L = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$ *has the desired properties.*

Proof. Surjectivity is clear.
Moreover,

$$\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_0 + a_1 + a_2 + a_3$$

and

$$a_0 + a_1 + a_2 + a_3 = 0 \Leftrightarrow a_3 = -a_0 - a_1 - a_2,$$

so that

$$\begin{aligned} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} &= \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ -a_0 - a_1 - a_2 \end{pmatrix} \\ &= a_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + a_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \end{aligned}$$

which finally shows that

$$\operatorname{Ker} l = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\rangle_{\mathbb{Z}} = \operatorname{Im} A.$$

□

Altogether we just showed that

$$\boxed{H_0^{\Delta}(\mathbb{CP}^1, \mathbb{Z}) \simeq H_0(C, \mathbb{Z}) \simeq \mathbb{Z}^4 / \operatorname{Im} A \simeq \mathbb{Z}.}$$

(iii) $\underline{H_1(C, \mathbb{Z})}$:

$H_1(C, \mathbb{Z}) = \operatorname{Ker} \delta_1' / \operatorname{Im} \delta_2' = \operatorname{Ker} A / \operatorname{Im} B$
In order to compute $\operatorname{Im} B$ it is enough to notice that the 1^{st} column of the matrix B can be obtained by linear combination of the remaining ones. If one agrees to denote the columns from left to right by c_1, c_2, c_3

and c_4, we have that $c_1 = c_2 + c_4 - c_3$. Moreover, the column vectors c_2, c_3 and c_4 are linearly independent which allows us to conclude that

$$\operatorname{Im} B = \langle c_2, c_3, c_4 \rangle_{\mathbb{Z}} = \left\langle \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle_{\mathbb{Z}}.$$

In order to compute $\operatorname{Ker} A$ we will this time need the reduced row echelon form of A which we denote by A''. Applying Gauss elimination we obtain the required matrix:

$$A'' = \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{aligned} & A'' \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \Leftrightarrow \quad & \begin{pmatrix} a_0 - a_3 - a_4 \\ a_1 + a_3 - a_5 \\ a_2 + a_4 + a_5 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \Leftrightarrow \quad & \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = a_3 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + a_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + a_5 \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \end{aligned}$$

so that

$$\operatorname{Ker} A = \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{Z}} = \operatorname{Im} B.$$

Altogether we just showed that

$$\boxed{H_1^{\Delta}(\mathbb{CP}^1,\mathbb{Z}) \simeq H_1(C,\mathbb{Z}) \simeq \text{Ker } A/\text{Im } B \simeq \{0\}.}$$

In order to compute the different homology groups $H_i^{\Delta}(\mathbb{CP}^1,\mathbb{Z})$ for $i = 0,1,2$ we worked with the chain complex

$$0 \to \mathbb{Z}^4 \xrightarrow{B} \mathbb{Z}^6 \xrightarrow{A} \mathbb{Z}^4 \to 0.$$

Now, in order to get the different cohomology groups $H^i_{\Delta}(\mathbb{CP}^1,\mathbb{Z})$ for $i = 0,1,2$ we need to consider the cochain complex

$$0 \to \Delta^0(\mathbb{CP}^1) \xrightarrow{d^0} \Delta^1(\mathbb{CP}^1) \xrightarrow{d^1} \Delta^2(\mathbb{CP}^1) \to 0,$$

which is by definition the cochain complex we obtain by dualization:

$$0 \to \text{Hom}(\Delta_0(\mathbb{CP}^1),\mathbb{Z}) \to \text{Hom}(\Delta_1(\mathbb{CP}^1),\mathbb{Z}) \to \text{Hom}(\Delta_2(\mathbb{CP}^1),\mathbb{Z}) \to 0.$$

Using the isomorphism between the chain groups we introduced above this is nothing else than

$$0 \to \text{Hom}(\mathbb{Z}^4,\mathbb{Z}) \to \text{Hom}(\mathbb{Z}^6,\mathbb{Z}) \to \text{Hom}(\mathbb{Z}^4,\mathbb{Z}) \to 0.$$

Now, applying the result $\text{Hom}(\mathbb{Z}^n,\mathbb{Z}) \simeq \mathbb{Z}^n$ we finally get the cochain complex

$$0 \to \mathbb{Z}^4 \xrightarrow{A^t} \mathbb{Z}^6 \xrightarrow{B^t} \mathbb{Z}^4 \to 0.$$

By performing similar computations as we did before for the different homology groups we get here that

$$H^0_{\Delta}(\mathbb{CP}^1,\mathbb{Z}) \simeq \mathbb{Z},$$
$$H^1_{\Delta}(\mathbb{CP}^1,\mathbb{Z}) \simeq \{0\},$$
$$H^2_{\Delta}(\mathbb{CP}^1,\mathbb{Z}) \simeq \mathbb{Z}.$$

Now, we need to take an interest into the cup product of the different cohomology groups $H^k_{\Delta}(\mathbb{CP}^1,\mathbb{Z}) \times H^l_{\Delta}(\mathbb{CP}^1,\mathbb{Z}) \xrightarrow{\smile} H^{k+l}_{\Delta}(\mathbb{CP}^1,\mathbb{Z})$. Note that we only have something non-zero in degree 0 and in degree 2 so that we only need to compute the cup product between two objects of degree 0 or between one object of degree 0 and one object of degree 2. Any other cup product will trivially give 0 as result because the corresponding target space will be $\{0\}$.

Let us first study the cup product involving two objects of degree 0. We have that

$$H^0_{\Delta}(\mathbb{CP}^1,\mathbb{Z}) \simeq \text{Ker } A^t = \mathbb{Z}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

so that a basis vector of $H^0_\Delta(\mathbb{CP}^1,\mathbb{Z})$ is given by $\left[\left(e^0_1\right)^* + \left(e^0_2\right)^* + \left(e^0_3\right)^* + \left(e^0_4\right)^*\right]$ where the $*$ denotes the dual vector. Let $\alpha,\beta \in H^0_\Delta(\mathbb{CP}^1,\mathbb{Z})$, *i.e.* $\alpha = a\left[\left(e^0_1\right)^* + \left(e^0_2\right)^* + \left(e^0_3\right)^* + \left(e^0_4\right)^*\right]$, $\beta = b\left[\left(e^0_1\right)^* + \left(e^0_2\right)^* + \left(e^0_3\right)^* + \left(e^0_4\right)^*\right]$ with $a,b \in \mathbb{Z}$.

$$\begin{aligned}
&\alpha \smile \beta \\
&= ab\left\{\left[\left(e^0_1\right)^* + \left(e^0_2\right)^* + \left(e^0_3\right)^* + \left(e^0_4\right)^*\right] \smile \left[\left(e^0_1\right)^* + \left(e^0_2\right)^* + \left(e^0_3\right)^* + \left(e^0_4\right)^*\right]\right\} \\
&= ab\left[\left(e^0_1\right)^* \smile \left(e^0_1\right)^* + \left(e^0_1\right)^* \smile \left(e^0_2\right)^* + \cdots + \left(e^0_4\right)^* \smile \left(e^0_3\right)^* + \left(e^0_4\right)^* \smile \left(e^0_4\right)^*\right].
\end{aligned}$$

However,

$$\begin{aligned}
\left[\left(e^0_i\right)^* \smile \left(e^0_j\right)^*\right]\left(e^0_k\right) &= \left[\left(e^0_i\right)^*\right]\left(e^0_k\right) \cdot \left[\left(e^0_j\right)^*\right]\left(e^0_k\right) \\
&= \delta_{ik}\cdot\delta_{jk} \\
&= \begin{cases} 1 & \text{if } i=j=k \\ 0 & \text{else} \end{cases},
\end{aligned}$$

so that

$$\begin{aligned}
&\alpha \smile \beta \\
&= ab\left[\left(e^0_1\right)^* \smile \left(e^0_1\right)^* + \left(e^0_2\right)^* \smile \left(e^0_2\right)^* + \left(e^0_3\right)^* \smile \left(e^0_3\right)^* + \left(e^0_4\right)^* \smile \left(e^0_4\right)^*\right] \\
&= ab\left[\left(e^0_1\right)^* + \left(e^0_2\right)^* + \left(e^0_3\right)^* + \left(e^0_4\right)^*\right].
\end{aligned}$$

All this shows that the cup product

$$\begin{aligned}
H^0_\Delta(\mathbb{CP}^1,\mathbb{Z}) \times H^0_\Delta(\mathbb{CP}^1,\mathbb{Z}) &\xrightarrow{\smile} H^0_\Delta(\mathbb{CP}^1,\mathbb{Z}) \\
\alpha,\beta &\longmapsto \alpha \smile \beta
\end{aligned}$$

can be identified with a map

$$\begin{aligned}
\mathbb{Z}\times\mathbb{Z} &\longrightarrow \mathbb{Z} \\
(a,b) &\longmapsto a\cdot b.
\end{aligned}$$

Now, let us study the cup product involving an object of degree 0 and an object of degree 2, *i.e.* we want to understand the map

$$\begin{aligned}
H^0_\Delta(\mathbb{CP}^1,\mathbb{Z}) \times H^2_\Delta(\mathbb{CP}^1,\mathbb{Z}) &\xrightarrow{\smile} H^2_\Delta(\mathbb{CP}^1,\mathbb{Z}) \\
\alpha,\beta &\longmapsto \alpha \smile \beta.
\end{aligned}$$

We have that

$$H^2_\Delta(\mathbb{CP}^1,\mathbb{Z}) \simeq \mathbb{Z}^4/\text{Im } B^t,$$

where

$$\text{Im } B^t = \left\langle \begin{pmatrix} 1\\0\\0\\1 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\-1 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\1 \end{pmatrix} \right\rangle_{\mathbb{Z}} \simeq \mathbb{Z}^3,$$

so that

$$H^2_\Delta(\mathbb{CP}^1,\mathbb{Z}) \simeq \mathbb{Z}^4/\text{Im } B^t \simeq \mathbb{Z} \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}.$$

This means that a basis vector of $H^2_\Delta(\mathbb{CP}^1,\mathbb{Z})$ is given by $\left[\left(e_4^2\right)^*\right]$. Let $\alpha \in H^0_\Delta(\mathbb{CP}^1,\mathbb{Z}), \beta \in H^2_\Delta(\mathbb{CP}^1,\mathbb{Z})$, *i.e.* $\alpha = a\left[\left(e_1^0\right)^* + \left(e_2^0\right)^* + \left(e_3^0\right)^* + \left(e_4^0\right)^*\right]$, $\beta = b\left[\left(e_4^2\right)^*\right]$ with $a, b, \in \mathbb{Z}$.

$$\begin{aligned}
&\alpha \smile \beta \\
&= ab\left\{\left[\left(e_1^0\right)^* + \left(e_2^0\right)^* + \left(e_3^0\right)^* + \left(e_4^0\right)^*\right] \smile \left[\left(e_4^2\right)^*\right]\right\} \\
&= ab\left[\left(e_1^0\right)^* \smile \left(e_4^2\right)^* + \left(e_2^0\right)^* \smile \left(e_4^2\right)^* + \left(e_3^0\right)^* \smile \left(e_4^2\right)^* + \left(e_4^0\right)^* \smile \left(e_4^2\right)^*\right].
\end{aligned}$$

But, (keep in mind the notations introduced at page 19)

$$\left[\left(e_i^0\right)^* \smile \left(e_4^2\right)^*\right]\left(e_j^2\right) = 0 \quad \text{if} \quad j \neq 4$$

and

$$\begin{aligned}
\left[\left(e_i^0\right)^* \smile \left(e_4^2\right)^*\right]\left(e_4^2\right) &= \left[\left(e_i^0\right)^*\right]\left(e_1^0\right) \cdot \left[\left(e_4^2\right)^*\right]\left(e_4^2\right) \\
&= \left[\left(e_i^0\right)^*\right]\left(e_1^0\right) \cdot 1 \\
&= \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{else} \end{cases},
\end{aligned}$$

so that

$$\begin{aligned}
\alpha \smile \beta &= ab\left[\left(e_1^0\right)^* \smile \left(e_4^2\right)^*\right] \\
&= ab\left(e_4^2\right)^*.
\end{aligned}$$

So again, all this shows that the cup product

$$\begin{aligned}
H^0_\Delta(\mathbb{CP}^1,\mathbb{Z}) \times H^2_\Delta(\mathbb{CP}^1,\mathbb{Z}) &\xrightarrow{\smile} H^2_\Delta(\mathbb{CP}^1,\mathbb{Z}) \\
\alpha, \beta &\longmapsto \alpha \smile \beta
\end{aligned}$$

can be identified with a map

$$\begin{array}{ccc} \mathbb{Z} \times \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ (a, b) & \longmapsto & a \cdot b. \end{array}$$

From the knowledge of $H^0_\Delta(\mathbb{CP}^1, \mathbb{Z}) \simeq \mathbb{Z}$, $H^1_\Delta(\mathbb{CP}^1, \mathbb{Z}) \simeq \{0\}$ and $H^2_\Delta(\mathbb{CP}^1, \mathbb{Z}) \simeq \mathbb{Z}$, we can conclude that

$$H^*(\mathbb{CP}^1) \simeq \mathbb{Z} \oplus \mathbb{Z} \simeq \mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot g,$$

where g takes the role of the generator of $H^2_\Delta(\mathbb{CP}^1, \mathbb{Z})$. Moreover, with the help of our computations around the cup product we know that $1 \smile g = g$ and $g \smile g = 0$. Since we only have non-zero elements in degree 0 and 2, we also know through Lemma 1.3.6 that our cohomology ring $H^*(\mathbb{CP}^1)$ is a commutative ring, so that it makes sense to consider the following map:

$$\begin{array}{ccc} \mathbb{Z}[x] & \xrightarrow{\varphi} & H^*(\mathbb{CP}^1) \\ x & \longmapsto & g. \end{array}$$

Obviously the map φ is surjective. By the above it is clear that $\varphi(x^2) = g \smile g = 0$ and from this we conclude that any polynomial with integer coefficients $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ will be mapped to $a_0 + a_1 g$ under the action of φ.
Consequently, $\varphi(f(x)) = 0 \Leftrightarrow a_0 + a_1 g = 0 \Leftrightarrow a_0 = 0$ and $a_1 = 0 \Leftrightarrow f(x) \in \left(x^2\right)$ which shows that Ker $\varphi = \left(x^2\right)$. Applying the first isomorphism Theorem for the map φ yields the isomorphism

$$\boxed{\mathbb{Z}[x] \big/ \left(x^2\right) \simeq H^*(\mathbb{CP}^1).}$$

1.4.2 The cohomology ring $H^*(\mathbb{CP}^n)$

The computations we have done in the preceding section can be generalized to any complex projective space of dimension n. This is exactly the content of the following Theorem:

Theorem 1.4.1.

$$H^*(\mathbb{CP}^n) \simeq \mathbb{Z}[x] \big/ \left(x^{n+1}\right).$$

A proof of this Theorem is given in section 3.2. of [7]. (cf. Theorem 3.19. p.220)

Remark 1.4.2. From the preceding Theorem we can conclude that:

$$\begin{aligned} H^{2i}(\mathbb{CP}^n) &\simeq \mathbb{Z} \quad \text{for } i = 0, \ldots, n, \\ H^{2i+1}(\mathbb{CP}^n) &\simeq \{0\} \quad \text{for } i = 0, \ldots, n-1. \end{aligned}$$

Remark 1.4.3. It is true that the computations we have done in the preceding section can be generalized to any complex projective space of dimension n. However, no explicit triangulation of $\mathbb{CP}^n$ is known for $n \geq 4$ even though we know that it has to exist. The interested reader can have a look at [16] for further details. Therefore, if we want to prove Theorem 1.4.1 we need to work with another cohomology theory than the one given by Δ-complexes. The proof of Theorem 1.4.1 which can be found in [7] is realized by means of CW-complexes. Compared to Δ-complexes these complexes are more laborious but the advantage is that one can easily compute with them, whereas Δ-complexes are more accessible but the price we have to pay is that the computations get quickly time-consuming as one can recognize in section 1.4.1.

Chapter 2

Chern Classes

2.1 Cohomology of Sheaves : Čech Cohomology

Let $\mathscr{F}$ be a sheaf of abelian groups on X, and $\underline{U} = \{U_\alpha\}$ a locally finite open cover. We define the following cochain groups:

$$\begin{aligned} C^0(\underline{U}, \mathscr{F}) &= \prod_{\alpha} \mathscr{F}(U_\alpha), \\ C^1(\underline{U}, \mathscr{F}) &= \prod_{\alpha \neq \beta} \mathscr{F}(U_\alpha \cap U_\beta), \\ &\vdots \\ C^p(\underline{U}, \mathscr{F}) &= \prod_{\alpha_0 \neq \alpha_1 \neq \cdots \neq \alpha_p} \mathscr{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}). \end{aligned}$$

An element $\sigma = \{\sigma_I \in \mathscr{F}\,(\cap U_{i_k})\}_{\#I=p+1}$ of $C^p(\underline{U}, \mathscr{F})$ is called a $p-$cochain of $\mathscr{F}$. We define a coboundary operator

$$d : C^p(\underline{U}, \mathscr{F}) \to C^{p+1}(\underline{U}, \mathscr{F})$$

by the formula

$$(d\sigma)_{i_0,\ldots,i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \sigma_{i_0,\ldots,\hat{i_j},\ldots,i_{p+1}} \Big|_{U_{i_0} \cap \cdots \cap U_{i_p}}.$$

In particular, if $\sigma = \{\sigma_U\} \in C^0(\underline{U}, \mathscr{F})$,

$$(d\sigma)_{U,V} = \sigma_V - \sigma_U$$

and if $\sigma = \{\sigma_{U,V}\} \in C^1(\underline{U}, \mathscr{F})$,

$$(d\sigma)_{U,V,W} = \sigma_{VW} - \sigma_{UW} + \sigma_{UV}$$

(omitting the restriction).

It is easy to see that $d^2 = 0$, so we get a cochain complex and we set

$$Z^p(\underline{U}, \mathscr{F}) = \text{Ker } d \subset C^p(\underline{U}, \mathscr{F})$$

and

$$H^p(\underline{U}, \mathscr{F}) = \frac{Z^p(\underline{U}, \mathscr{F})}{dC^{p-1}(\underline{U}, \mathscr{F})}.$$

Now, given two coverings $\underline{U} = \{U_\alpha\}_{\alpha \in I}$ and $\underline{U'} = \{U'_\beta\}_{\beta \in I'}$ of X, we say that $\underline{U'}$ is a **refinement** of $\underline{U}$ if for every $\beta \in I'$ there exists $\alpha \in I$ such that $U'_\beta \subset U_\alpha$. We write $\underline{U'} < \underline{U}$.

We define the p^{th} **Čech cohomology group** of $\mathscr{F}$ on X to be the direct limit of the $H^p(\underline{U}, \mathscr{F})$'s as $\underline{U}$ becomes finer and finer:

$$H^p(X, \mathscr{F}) \;=\; \varinjlim_{\underline{U}} \; H^p(\underline{U}, \mathscr{F}).$$

When there is a possibility of confusion, we will denote Čech cohomology groups by $\check{H}$. Clearly, for any covering $\underline{U}$

$$H^0(X, \mathscr{F}) = H^0(\underline{U}, \mathscr{F}) = \mathscr{F}(X).$$

Note that if $X \subset Y$ is a closed subspace, $\mathscr{F}$ any sheaf of abelian groups on X, then extending $\mathscr{F}$ by zero to a sheaf on Y, we have

$$H^p(X, \mathscr{F}) = H^p(Y, \mathscr{F}).$$

The definition of $H^p(X, \mathscr{F})$ as a direct limit is, in practice, more or less impossible to work with. What is needed is a simple sufficient condition on a cover $\underline{U}$ for

$$H^p(\underline{U}, \mathscr{F}) = H^p(X, \mathscr{F}),$$

which is provided by the following Theorem:

Leray Theorem. *If the covering $\underline{U}$ is acyclic for the sheaf $\mathscr{F}$ in the sense that*

$$H^q(U_{i_1} \cap \cdots \cap U_{i_p}, \mathscr{F}) = 0, \qquad q > 0, \quad \textit{any} \quad i_1, \ldots, i_p,$$

then $H^p(\underline{U}, \mathscr{F}) \simeq H^p(X, \mathscr{F})$.

The interested reader may find a proof of this Theorem at page 40 of [4].

2.2 The Picard Group of a Complex Manifold X

During this whole section every line bundle is taken to be holomorphic. Recall that for any holomorphic line bundle $L \xrightarrow{\pi} X$ on the complex manifold X, we can find an open cover $\{U_\alpha\}$ of X and trivializations

$$\varphi_\alpha : L_{U_\alpha} \to U_\alpha \times \mathbb{C}$$

of $L_{U_\alpha} = \pi^{-1}(U_\alpha)$. The transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_1(\mathbb{C})$ for L relative to the trivializations $\{\varphi_\alpha\}$ are defined by

$$g_{\alpha\beta}(z) = \left(\varphi_\alpha \circ \varphi_\beta^{-1}\right)\Big|_{L_z} \in GL_1(\mathbb{C}).$$

The functions $g_{\alpha\beta}$ are clearly holomorphic, nonvanishing and moreover they satisfy:

$$(1) \qquad \begin{cases} g_{\alpha\beta} \cdot g_{\beta\alpha} & = 1 \\ g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} & = 1. \end{cases}$$

Conversely, given a collection of functions $\{g_{\alpha\beta} \in \mathcal{O}_X^*(U_\alpha \cap U_\beta)\}$ satisfying these identities, we can construct a line bundle L with transition functions $\{g_{\alpha\beta}\}$ by taking the union of $U_\alpha \times \mathbb{C}$ over all α and identifying $\{z\} \times \mathbb{C}$ in $U_\alpha \times \mathbb{C}$ and $U_\beta \times \mathbb{C}$ via multiplication by $g_{\alpha\beta}(z)$. Now, for an arbitrary holomorphic line bundle L, for any collection of nonzero holomorphic functions $f_\alpha \in \mathcal{O}_X^*(U_\alpha)$ we can define alternate trivializations of L over $\{U_\alpha\}$ by

$$\varphi'_\alpha = f_\alpha \cdot \varphi_\alpha,$$

so that transition functions $g'_{\alpha\beta}$ for L relative to $\{\varphi'_\alpha\}$ will then be given by

$$(2) \quad g'_{\alpha\beta} = \varphi'_\alpha \circ \varphi'^{-1}_\beta = f_\alpha \cdot \varphi_\alpha \circ f_\beta^{-1} \cdot \varphi_\beta^{-1} = f_\alpha \cdot f_\beta^{-1} \cdot \varphi_\alpha \circ \varphi_\beta^{-1} = \frac{f_\alpha}{f_\beta} \cdot g_{\alpha\beta}.$$

On the other hand, we see that any other trivialization of L over $\{U_\alpha\}$ can be obtained in this way. Finally, this allows us to conclude that collections $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ of transition functions define the same line bundle if and only if there exist functions $f_\alpha \in \mathcal{O}_X^*(U_\alpha)$ satisfying (2).
The description of line bundles by transition functions provides a sheaf theoretic interpretation of them. First, we notice that the transition functions $\{g_{\alpha\beta} \in \mathcal{O}_X^*(U_\alpha \cap U_\beta)\}$ for a line bundle $L \xrightarrow{\pi} X$ represent a Čech 1-cochain on X with coefficients in $\mathcal{O}_X^*$; the relation (1) asserts that $d(\{g_{\alpha\beta}\}) = 0$, *i.e.* $\{g_{\alpha\beta}\}$ is a Čech cocycle. Moreover, from the last paragraph we conclude that two cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ define the same line bundle if and only if their difference $\left\{g_{\alpha\beta} \cdot g'^{-1}_{\alpha\beta}\right\}$ is a Čech coboundary.

Consequently, the set of isomorphism classes of line bundles on X is just $H^1(X, \mathcal{O}_X^*)$.

We can give the set of isomorphism classes of line bundles on X the structure of a group, multiplication being given by tensor product and inverses by dual bundles. If L is given by transition functions $\{g_{\alpha\beta}\}$, L' by $\{g'_{\alpha\beta}\}$, we have that

$$L \otimes L' \sim \left\{g_{\alpha\beta} g'_{\alpha\beta}\right\}, \quad L^* \sim \left\{g_{\alpha\beta}^{-1}\right\},$$

and so under the identification described above, the group structure on the set of isomorphism classes of line bundles is the same (means we can identify both by an isomorphism) as the group structure on $H^1(X, \mathcal{O}_X^*)$. The group $H^1\left(X, \mathcal{O}_X^*\right)$ is called the **Picard group of** X, and is denoted by $\mathrm{Pic}(X)$.

Remark 2.2.1. (Based on [9].) There exists also the notion of a Picard group for a compact Riemann surface X. It is defined as follows: Let X be a compact Riemann surface. Let $\mathrm{Div}(X)$ denote the free abelian group generated by points of X. This is the set of divisors on X. Any element $D \in \mathrm{Div}(X)$ can be written as

$$D = \sum_{x \in X} n_x \cdot x,$$

where $n_x \in \mathbb{Z}$ and only finitely many of them are different from zero. On $\mathrm{Div}(X)$ we can consider the notion of a principal divisor corresponding to a meromorphic function f. More precisely, for a global meromorphic function f on X, *i.e.* $f \in \mathcal{M}(X)$ we set

$$(f) := \sum_{x \in X} (\mathrm{ord}_x f) \cdot x \in \mathrm{Div}(X),$$

where the order of f is defined as follows:

$$\mathrm{ord}_a f := \begin{cases} \mathrm{mult}_a f, & f(a) = 0 \\ -\mathrm{mult}_a f, & f(a) = \infty \\ 0, & \text{else} \end{cases}$$

and $\mathrm{mult}_a f$ is determined by means of the $1:1$ correspondence
$\mathcal{M}(X) \leftrightarrow \{\text{holomorphic maps } X \to \hat{\mathbb{C}} \text{ not identically } \infty\}$ and the local behaviour Theorem of holomorphic maps of Riemann surfaces.

Theorem (Local behaviour of holomorphic maps). *(Theorem 3.1 of [9].) Let X, Y be Riemann surfaces. Let $g : X \to Y$ be a holomorphic map. Let $a \in X$, $b := g(a) \in Y$. Then there exists an integer $k \geq 0$ such that locally around a the morphism g looks as*

$$z \mapsto z^k,$$

i.e., there exists a chart $U \xrightarrow{\varphi} V$, $a \in U$, $\varphi(a) = 0$, and a chart $U' \xrightarrow{\psi} V'$, $b \in U'$, $\psi(b) = 0$, such that $g(U) \subset U'$ and $\psi \circ g|_U \circ \varphi^{-1}(z) = z^k$.

The number k of the previous Theorem is uniquely determined for a given holomorphic map g and a given point $a \in X$. Using the $1:1$ correspondence we can associate to f a holomorphic map $g : X \to \hat{\mathbb{C}}$ which is not identically ∞ and set $\text{mult}_a f := \text{mult}_a g$.
We call (f) the principal divisor corresponding to the meromorphic function f. For $f, g \in \mathcal{M}(X)$, $x \in X$ we have $\text{mult}_x(f \cdot g) = \text{mult}_x f + \text{mult}_x g$, so that

$$\begin{aligned} (f \cdot g) &= (f) + (g), \\ \left(\frac{1}{f}\right) &= -(f), \end{aligned}$$

which shows that the set of all principal divisors $\text{PDiv}(X)$ is a subgroup of $\text{Div}(X)$. Since we work in an abelian context it makes sense to consider the following quotient group

$$\text{Pic}(X) = {}^{\text{Div}(X)}\!/_{\text{PDiv}(X)},$$

which is called the **Picard group of** X. It turns out that this one is isomorphic to the group $H^1(X, \mathcal{O}_X^*)$, so that both notions are equivalent. (For more details have a look at [4] p.161 and use the fact that any compact Riemann surface can be embedded into complex projective 3-space.)

2.3 The First Chern Class of Line Bundles

Let X be a complex manifold of dimension n.

Definition 2.3.1 (**First Chern Class**)**.** The exact sequence of sheaves on X

$$0 \to \underline{\mathbb{Z}} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$

gives a coboundary map (Proposition 1.3.4 of [17])

$$H^1(X, \mathcal{O}_X^*) \xrightarrow{d} H^2(X, \mathbb{Z}).$$

For an isomorphism class of line bundles $L \in \text{Pic}(X) = H^1(X, \mathcal{O}_X^*)$, we define the **first Chern class** $c_1(L)$ of L (or simply Chern class) to be $d(L) \in H^2(X, \mathbb{Z})$.

Remark 2.3.2. By $\underline{\mathbb{Z}}$ we denote the constant sheaf, *i.e.* $\underline{\mathbb{Z}}(U) = \mathbb{Z}$ for any open set $U \subset X$. In order to get the coboundary map described in the previous definition we apply the result that any short exact sequence of sheaves induces a long exact sequence in cohomology. (Proposition 1.3.4 of [17].) Moreover, we also use here the fact that $H^2(X, \underline{\mathbb{Z}}) \simeq H^2(X, \mathbb{Z})$. (The interested reader may find a proof of this fact at page 42 of [4].)

Let us also try to understand why the given sequence of sheaves is exact: In order to show this, it is sufficient to check it on stalks. (Below, we use

index notation to characterize the stalk at a.)
It is clear that the map $\alpha : \mathbb{Z}_a \to \mathcal{O}_{X,a}$, $n \mapsto 2\pi i n$ is injective. Moreover, for $f \in \mathcal{O}_{X,a}$ we have that $\exp f = 1 \Leftrightarrow f = 2\pi i n$ so that Ker exp = Im α. It remains to show that the map $\exp : \mathcal{O}_{X,a} \to \mathcal{O}^*_{X,a}$ is surjective, *i.e.* for $f \in \mathcal{O}^*_{X,a}$ one needs to find $g \in \mathcal{O}_{X,a}$ such that $\exp g = f$. In order to solve this equation we use the identification $\mathcal{O}_{X,a} \simeq \mathbb{C}\{x_1, \ldots, x_n\}$, *i.e.* the set of equivalence classes of holomorphic functions defined in a neighbourhood around a is isomorphic to the set of convergent power series in $x_1, \ldots, x_n$. Similarly, the set of equivalence classes of invertible holomorphic functions defined in a neighbourhood around a is isomorphic to the set of convergent power series in $x_1, \ldots, x_n$ with non-zero constant term. With help of this identification and the power series expression of the exponential map, the before mentioned equation results in constructing $g = \sum_{i=0}^{\infty} a_i x^i$ with $a_i \in \mathbb{C}$ such that $\exp g = \sum_{i=0}^{\infty} \frac{1}{i!} g^i = f$ for a given $f = \sum_{i=0}^{\infty} b_i x^i$ with $b_i \in \mathbb{C}$ for $i \geq 1$ and $b_0 \neq 0$. This final equation admits a solution which one obtains by identification of the coefficients.

Remark 2.3.3. As an immediate consequence of the definition, note that

$$c_1(L \otimes L') = c_1(L) + c_1(L')$$

and

$$c_1(L^*) = -c_1(L).$$

Also, if $f : X \to Y$ is a holomorphic map of complex manifolds, the diagram

$$\begin{array}{ccc} H^1(X, \mathcal{O}^*_X) & \longrightarrow & H^2(X, \mathbb{Z}) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ H^1(Y, \mathcal{O}^*_Y) & \longrightarrow & H^2(Y, \mathbb{Z}) \end{array}$$

commutes, so that for $L \xrightarrow{\pi} Y$ any line bundle,

$$c_1(f^* L) = f^* c_1(L).$$

Example 2.3.4. In section 1.4.1 we have established that $H^2_{\Delta}(\mathbb{CP}^1, \mathbb{Z}) \simeq \mathbb{Z}$. Hence, the first Chern class of a line bundle $L \in \mathrm{Pic}(\mathbb{CP}^1)$ is just an integer. In particular since d is a morphism, we immediately conclude that the first Chern class of the trivial line bundle I over the complex projective space of dimension one is 0, *i.e.* $c_1(I) = d(I) = 0$.
Notice that the last Remark holds true for the trivial line bundle on any complex manifold X, in the sense that its first Chern class is just the corresponding 0 class in $H^2(X, \mathbb{Z})$.

2.4 Chern Classes of vector bundles of rank r

In what follows we will use the axiomatic approach of *Grothendieck* [5] in order to define the ith Chern class $c_i(E)$ of an arbitrary vector bundle E of rank r. This method presents the advantage that one can obtain the different Chern classes from the line bundle case.

In order to understand the explicit definition, we need the following essential considerations around the *Leray-Hirsch* Theorem:
Let E be a vector bundle of rank r on a complex manifold X. Consider the corresponding projective bundle $\mathbb{P}(E) \xrightarrow{\pi} X$, *i.e.* locally one has that $\mathbb{P}(E) \simeq X \times \mathbb{CP}^{r-1}$. The bundle projection π and the cohomology functor induce the following homomorphism of rings:

$$\pi^* : H^*(X, \mathbb{Z}) \to H^*(\mathbb{P}(E), \mathbb{Z}),$$

which naturally endows $H^*(\mathbb{P}(E), \mathbb{Z})$ with an $H^*(X, \mathbb{Z})$-**module structure**. Consider also the exact sequence of sheaves on $\mathbb{P}(E)$:

$$0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_{\mathbb{P}(E)} \xrightarrow{\exp} \mathcal{O}^*_{\mathbb{P}(E)} \to 0.$$

The induced long exact sequence in cohomology provides a homomorphism

$$H^1\left(\mathbb{P}(E), \mathcal{O}^*_{\mathbb{P}(E)}\right) \xrightarrow{c_1} H^2(\mathbb{P}(E), \mathbb{Z}).$$

Use this homomorphism to set $\xi = c_1\left([\mathcal{O}_{\mathbb{P}(E)}(1)]\right)$, where $\mathcal{O}_{\mathbb{P}(E)}(1)$ is the line bundle dual to the tautological bundle $\mathscr{S}$ on $\mathbb{P}(E)$ defined as follows:
Pull back E to $\mathbb{P}(E)$ along $\pi : \mathbb{P}(E) \to X$ and obtain
$\pi^*E = \mathbb{P}(E) \underset{X}{\times} E := \{\, (x, v) \in \mathbb{P}(E) \times E \mid \pi(x) = p(v) \}$:

$$\begin{array}{ccc} \pi^*E = \mathbb{P}(E) \underset{X}{\times} E \subset \mathbb{P}(E) \times E & \longrightarrow & E \\ \downarrow & & \downarrow p \\ \mathbb{P}(E) & \xrightarrow[\pi]{} & X. \end{array}$$

Define $\mathscr{S} \subset \pi^*E$ by $\mathscr{S} = \{\, (z, v) \in \pi^*E \mid v \in z \}$. Equivalently one can also say: $\xi = -c_1([\mathscr{S}])$.

Claim. *π^*E is a vector bundle of rank r on $\mathbb{P}(E)$.*

Sketch of the proof. Let $\{g_{ij}\}$ be a cocycle defining E with respect to a trivialization $\cup U_i = X$, *i.e.* $g_{ij} : U_i \cap U_j \to GL_r(\mathbb{C})$. We have that $\cup \pi^{-1}(U_i) = \mathbb{P}(E)$ and a cocycle of π^*E can be given by:

$$g'_{ij} := \pi^*(g_{ij}) = g_{ij} \circ \pi : \pi^{-1}U_i \cap \pi^{-1}U_j \xrightarrow{\pi} U_i \cap U_j \xrightarrow{g_{ij}} GL_r(\mathbb{C}).$$

This shows that π^*E is a vector bundle of rank r on $\mathbb{P}(E)$. □

Claim. *$\mathscr{S}$ is a vector line subbundle of π^*E.*

Sketch of the proof. We will establish how one looks locally at the vector bundle $\mathscr{S}$. Let $(z, v) \in \mathscr{S}$. Then $\pi(z) = x$ for some $x \in X$, *i.e.* $z \in \pi^{-1}(x) = \mathbb{P}\left(p^{-1}(x)\right) = \mathbb{P}(E_x)$ and $v \in E_x$ since we have that $v \in z$. Moreover, we know that locally $E_{U_i} \simeq U_i \times \mathbb{C}^r$ and $\mathbb{P}(E_{U_i}) \simeq U_i \times \mathbb{CP}^{r-1}$ so that we get the following local diagram:

$$\begin{array}{ccc} \pi^*E_{U_i} \subset\ U_i \times \mathbb{CP}^{r-1} \times U_i \times \mathbb{C}^r & \longrightarrow & U_i \times \mathbb{C}^r \\ \downarrow & & \downarrow \\ U_i \times \mathbb{CP}^{r-1} & \longrightarrow & U_i, \end{array}$$

i.e. $\mathscr{S}_{U_i} \subset \pi^*E_{U_i} \simeq U_i \times \mathbb{CP}^{r-1} \times \mathbb{C}^r$, so locally one has that $\mathscr{S}_{U_i} \simeq \{(x, l, v) \in U_i \times \mathbb{CP}^{r-1} \times \mathbb{C}^r \mid v \in l\}$. Consider now some tuple $(x, l, v) \in \mathscr{S}_{U_i}$, *i.e.* $l = \langle l_0, \ldots, l_{r-1} \rangle$ and $v = (v_0, \ldots, v_{r-1})$. Look at the standard chart $\varphi_0 : V_0 \subset \mathbb{CP}^{r-1} \to \mathbb{C}^{r-1}$ of $\mathbb{CP}^{r-1}$ given by $l_0 = 1$. The condition that $v \in l$ yields that the 2×2 minors of the following matrix are zero

$$\begin{pmatrix} l_0 & l_1 & \cdots & l_{r-1} \\ v_0 & v_1 & \cdots & v_{r-1} \end{pmatrix},$$

which results in the following system of equations

$$(1) \qquad \begin{cases} v_1 &= l_1 v_0 \\ v_2 &= l_2 v_0 \\ & \vdots \\ v_{r-1} &= l_{r-1} v_0. \end{cases}$$

This allows us to construct the following isomorphism:

$$\begin{array}{c} \mathscr{S}_{U_i, V_0} \subset U_i \times V_0 \times \mathbb{C}^r \xrightarrow{\text{iso}} U_i \times \mathbb{C}^{r-1} \times \mathbb{C}^r \\ (x, \varphi_0(l), (v_0\,, \ldots\,, v_{r-1})) \mapsto (x, \varphi_0(l), (v_0\,, v_1 - l_1 v_0\,, \ldots\,, v_{r-1} - l_{r-1} v_0))\,. \end{array}$$

Combining this with (1) induces the isomorphism we are looking for namely

$$\begin{array}{rcl} \mathscr{S}_{U_i, V_0} & \xrightarrow{\text{iso}} & U_i \times \mathbb{C}^{r-1} \times \mathbb{C} \\ (x, \varphi_0(l), (v_0\,, \ldots\,, v_{r-1})) & \mapsto & (x, \varphi_0(l), v_0)\,. \end{array}$$

In order to finish the proof one also needs to check the cocycle condition. However, this is immediate as we have just shown that $\mathscr{S}$ is a submanifold of π^*E which is a vector bundle of rank r on $\mathbb{P}(E)$ and each fiber of $\mathscr{S}$ is a vector subspace of the corresponding fiber of π^*E. Hence, this allows us to extract by restriction a set of cocycles for $\mathscr{S}$ which then obviously satisfy the cocycle condition. □

We are now ready to state the *Leray-Hirsch* Theorem in this context:

Theorem 2.4.1 (**Leray-Hirsch**). *$H^*(\mathbb{P}(E),\mathbb{Z})$ is a free $H^*(X,\mathbb{Z})$-module generated by $1,\xi,\xi^2,\dots\xi^{r-1}$, i.e. every $a\in H^*(\mathbb{P}(E),\mathbb{Z})$ can be written in a unique way as $a=a_0\cdot 1+a_1\cdot\xi+\cdots+a_{r-1}\cdot\xi^{r-1}$ with $a_i\in H^*(X,\mathbb{Z})$.*

Remark 2.4.2. We view $H^*(\mathbb{P}(E),\mathbb{Z})$ as a module over the ring $H^*(X,\mathbb{Z})$ by defining scalar multiplication by $a\cdot\xi=\pi^*(a)\smile\xi$, for $a\in H^*(X,\mathbb{Z})$ and $\xi\in H^*(\mathbb{P}(E),\mathbb{Z})$.
The interested reader can find a proof of the *Leray-Hirsch* Theorem in [4] p. 606 or in [10] Theorem 2.1.2 p.30-31.

Now, consider $\xi^r=\underbrace{\xi\smile\cdots\smile\xi}_{r\text{ times}}\in H^{2r}(\mathbb{P}(E),\mathbb{Z})$. Using the *Leray-Hirsch* Theorem we know that ξ^r can be written in a unique way as

$$\xi^r=b_0\cdot 1+b_1\cdot\xi+\cdots+b_{r-1}\cdot\xi^{r-1}, \tag{1}$$

where $b_i\in H^*(X,\mathbb{Z})$ and $\cdot$ denotes the scalar-multiplication we have in $H^*(\mathbb{P}(E),\mathbb{Z})$. Let us have a closer look at the coefficients $b_i\in H^*(X,\mathbb{Z})=\oplus H^k(X,\mathbb{Z})$. Any single one can be decomposed in a sum $b_i=\sum b_{ij}$ where $b_{ij}\in H^j(X,\mathbb{Z})$. Hence, $b_i=\sum b_{i2(r-i)}+\sum_{j\neq 2(r-i)}b_{ij}$. From this it follows that $\xi^r=\sum b_i\xi^i=\sum b_{i2(r-i)}\cdot\xi^i+\sum_{j\neq 2(r-i)}b_{ij}\xi^i$. Note that the first sum lies in $H^{2r}(\mathbb{P}(E),\mathbb{Z})$ whereas the second sum does not have any term in degree $2r$. From this we conclude that $\sum_{j\neq 2(r-i)}b_{ij}\xi^i=0$, *i.e.* we just showed that $\sum_{j\neq 2(r-i)}b_{ij}=0,\forall i$ which allows us to conclude that $b_i\in H^{2(r-i)}(X,\mathbb{Z})$.

These considerations allow us now to define rigorously the **ith Chern class** of a vector bundle of rank r:

Definition 2.4.3. Let E be a vector bundle of rank r on a complex manifold X of dimension n. For each $i=0,1,\dots,r$ we define the **ith Chern class** $c_i(E)\in H^{2i}(X,\mathbb{Z})$ by the requirement $c_0(E)=1$ and

$$\sum_{i=0}^{r}c_i(E)\cdot\xi^{r-i}=0$$

in $H^{2r}(\mathbb{P}(E),\mathbb{Z})$, *i.e.* $c_i(E):=-b_{r-i}$ (using the unique decomposition given by equation (1)).

For convenience we define the **total Chern class**

$$c(E)=c_0(E)+c_1(E)+\cdots+c_r(E)$$

and the **Chern polynomial**

$$c_t(E)=c_0(E)+c_1(E)t+\cdots+c_r(E)t^r.$$

Remark 2.4.4. Chern polynomials present the big advantage that they keep track of the degree of the Chern class during computations so that no confusion is possible. This is very useful when we try to compute the different Chern classes of a given vector bundle by means of Chern classes of other vector bundles we already know. (Have a look at axiom A2 hereafter.) We will feel the power of this tool with help of the examples given here below. Moreover, notice that by definition, for a vector bundle E of rank r on a complex manifold X of dimension n, we only consider $r+1$ Chern classes $c_i(E) \in H^{2i}(X, \mathbb{Z})$ for $i = 0, 1, \ldots, r$. For $i > r$ we have that $c_i(E) = 0$ (cf. Theorem 2.4.1). From this we immediately conclude that for an arbitrary line bundle L on X we only consider its 0-Chern class which is by default set to be 1 and its first Chern class. In other words the Chern polynomial of an arbitrary line bundle L on X is given by $c_t(L) = c_0(L) + c_1(L)t = 1 + c_1(L)t$. Finally, observe that any Chern class $c_i(E) \in H^{2i}(X, \mathbb{Z})$ lies in a cohomology group of even degree, *i.e.* according to Theorem 1.3.6 the cup product between arbitrary Chern classes is commutative.

Axiomatic approach of Grothendieck (Based on [6] p.430)**.**

A1. (Naturality) If $f : X \to Y$ is a holomorphic map of complex manifolds, and E is a vector bundle of rank r on Y, then for each i

$$c_i(f^*E) = f^*c_i(E).$$

A2. (Additivity) If $0 \to E' \to E \to E'' \to 0$ is an exact sequence of vector bundles on X, then

$$c_t(E) = c_t(E') \cdot c_t(E'').$$

A3. (Normalization) If $E = L$ is some line bundle on X. Then $\mathbb{P}(L) = X$, and $\mathcal{O}_{\mathbb{P}(L)}(1) = \mathcal{O}_X(1) = L^*$, *i.e.* $\xi = c_1(L^*)$.
Indeed, by definition:

$$\begin{aligned} c_0(L) \cdot \xi + c_1(L) \cdot 1 = 0 \quad &\Leftrightarrow \quad \xi + c_1(L) = 0 \\ &\Leftrightarrow \quad \xi = -c_1(L) = c_1(L^*). \end{aligned}$$

Remark 2.4.5. There are several different approaches to define Chern classes. Here in this paper we gave the definition with help of the *Leray-Hirsch* Theorem, which immediately ensures uniqueness. Analogously, one could also have defined Chern classes by the preceding three axioms. This is then well defined, since any two theories which respect these three axioms are actually equal and so give rise to the same Chern classes. (The interested reader may find more details about this in [12] Theorem 5.12.)

Note that the previous three axioms are actually just consequences of Definition 2.4.3. In fact, axiom A3 is a direct application of it and shows that it is compatible with Definition 2.3.1 of the first Chern class of line bundles which we already met in section 2.3. For axiom A1 (cf. section 5.4 in [12].), observe that a holomorphic map $f : X \to Y$ induces a fiber bundle map $u : \mathbb{P}(f^*E) \to \mathbb{P}(E)$ which makes the following diagram commute:

$$\begin{array}{ccc} \mathbb{P}(f^*E) & \xrightarrow{u} & \mathbb{P}(E) \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow{f} & Y. \end{array}$$

From this diagram one gets the following commutative diagram:

$$\begin{array}{ccc} H^*(\mathbb{P}(f^*E)) & \xleftarrow{u^*} & H^*(\mathbb{P}(E)) \\ {\scriptstyle \pi_X^*}\uparrow & & \uparrow{\scriptstyle \pi_Y^*} \\ H^*(X) & \xleftarrow{f^*} & H^*(Y). \end{array}$$

Thus, when applying u^* to both sides of the equality $\sum_{i=0}^{r} c_i(E) \cdot \xi_E^{r-i} = 0$ we obtain that

$$\begin{aligned} & u^*\left(\sum_{i=0}^{r} c_i(E) \cdot \xi_E^{r-i}\right) = 0 \;\; u^*(0) = 0 \text{ as } u^* \text{ is a ring hom.} \\ \Leftrightarrow \quad & u^*\left(\sum_{i=0}^{r} \pi_Y^*(c_i(E)) \cdot \xi_E^{r-i}\right) = 0 \;\; \text{cf. Remark 2.4.2} \\ \Leftrightarrow \quad & \sum_{i=0}^{r} u^*[\pi_Y^*(c_i(E))] \cdot u^*(\xi_E^{r-i}) = 0 \;\; u^* \text{ is a homomorphism of rings} \\ \Leftrightarrow \quad & \sum_{i=0}^{r} \pi_X^*(f^*c_i(E)) \cdot u^*(\xi_E^{r-i}) = 0 \;\; \text{comm. of the above diagram} \\ \Leftrightarrow \quad & \sum_{i=0}^{r} \pi_X^*(f^*c_i(E)) \cdot (u^*(\xi_E))^{r-i} = 0 \;\; u^* \text{ is a homomorphism of rings} \\ \Leftrightarrow \quad & \sum_{i=0}^{r} \pi_X^*(f^*c_i(E)) \cdot \xi_{f^*E}^{r-i} = 0 \;\; \text{we use } u^*(\xi_E) \overset{(1)}{=} \xi_{f^*E} \\ \Leftrightarrow \quad & \sum_{i=0}^{r} f^*c_i(E) \cdot \xi_{f^*E}^{r-i} = 0 \;\; \text{cf. Remark 2.4.2} \end{aligned}$$

(1): Let us denote by $\mathscr{S}_Y$ the tautological bundle on $\mathbb{P}(E)$ and by $\mathscr{S}_X$ the tautological bundle on $\mathbb{P}(f^*E)$. We know that $\xi_E := c_1([\mathscr{S}_Y^*])$ and

that $\xi_{f^*E} := c_1([\mathscr{S}_X^*])$. According to Remark 2.3.3 the following diagram commutes:

$$\begin{array}{ccc} [\mathscr{S}_Y^*] \in H^1\left(\mathbb{P}(E), \mathcal{O}^*_{\mathbb{P}(E)}\right) & \xrightarrow{c_1} & H^2\left(\mathbb{P}(E), \mathbb{Z}\right) \ni \xi_E \\ \Big\downarrow{\scriptstyle u^*} & & \Big\downarrow{\scriptstyle u^*} \\ [\mathscr{S}_X^*] \in H^1\left(\mathbb{P}(f^*E), \mathcal{O}^*_{\mathbb{P}(f^*E)}\right) & \xrightarrow{c_1} & H^2\left(\mathbb{P}(f^*E), \mathbb{Z}\right) \ni \xi_{f^*E}. \end{array}$$

By definition of the tautological bundle we have that $\mathscr{S}_X = u^*\mathscr{S}_Y$. This, together with the commutativity of the preceding diagram, allows us to conclude that $u^*(\xi_E) = \xi_{f^*E}$.
Moreover, by definition, we also have that

$$\sum_{i=0}^{r} c_i(f^*E) \cdot \xi_{f^*E}^{r-i} = 0,$$

so that together with the *Leray-Hirsch* Theorem one comes to the conclusion that

$$c_i(f^*E) = f^*c_i(E).$$

For axiom A2 the interested reader can find the details of the computations in section 5.6 of [12].

Remark/Example 2.4.6. Imagine that there is an exact sequence of vector bundles $0 \to E' \to E \to E'' \to 0$ such that $c_t(E') = 1 + c_1 t$ and $c_t(E'') = 1 + c_2 t^2$. Then $c_t(E) = c_t(E') \cdot c_t(E'') = (1 + c_1 t) \cdot (1 + c_2 t^2) = 1 + c_1 t + c_2 t^2 + c_1 c_2 t^3$ by A2. Here it is important to always keep in mind that the products involved are actually cup products. By simplicity we will denote these cup products by simple multiplication symbol instead of $\smile$.
Finally, as we have seen in Remark 2.4.4, the cup product is commutative for Chern classes, *i.e.* here we have that $c_t(E) = c_t(E') \cdot c_t(E'') = c_t(E'') \cdot c_t(E')$.

Example 2.4.7. Let I be the trivial line bundle on X, and let us consider the exact sequence $0 \to I \to I \oplus I \to I \to 0$. In Remark 2.4.4 we have already seen that $c_t(I) = 1 + c_1(I)t$. Using axiom A2 we get that $c_t(I \oplus I) = c_t(I) \cdot c_t(I) = (1 + c_1(I)t) \cdot (1 + c_1(I)t) = 1 + c_1(I)t + c_1(I)t + c_1(I) \cdot c_1(I)t^2 = 1 + 2c_1(I)t + c_1(I) \cdot c_1(I)t^2$. However, as we have established in Example 2.3.4 we know that $c_1(I) = 0$, so that finally $c_t(I) = 1$. In other words we conclude from this that $c_0(I \oplus I) = 1, c_1(I \oplus I) = 0, c_2(I \oplus I) = 0$ and consequently we get that $c(I \oplus I) = 1$.

Example 2.4.8. Let L_1 and L_2 be two line bundles on X, and let us consider the exact sequence $0 \to L_1 \to L_1 \oplus L_2 \to L_2 \to 0$. We know that $c_t(L_1) = 1 + c_1(L_1)t$ and that $c_t(L_2) = 1 + c_1(L_2)t$. Using axiom A2 we get that

$c_t(L_1 \oplus L_2) = c_t(L_1) \cdot c_t(L_2) = 1 + c_1(L_2)t + c_1(L_1)t + c_1(L_1) \cdot c_1(L_2)t^2$. From this we conclude that $c_0(L_1 \oplus L_2) = 1$, $c_1(L_1 \oplus L_2) = c_1(L_1) + c_2(L_2)$ and $c_2(L_1 \oplus L_2) = c_1(L_1) \cdot c_1(L_2)$ so that the total Chern class is given by $c(L_1 \oplus L_2) = 1 + c_1(L_1) + c_2(L_2) + c_1(L_1) \cdot c_1(L_2)$.
Similarly, one can repeat these computations for an arbitrary extension $0 \to L_1 \to E \to L_2 \to 0$ and get the same result. From this one concludes that Chern classes do not see different extensions in the sense that they do not allow to distinguish them.

Example 2.4.9. We can extend by induction the arguments of the preceding Example in order to compute the Chern classes of the vector bundle E of rank n defined by the direct sum of n different line bundles $L_1, \ldots, L_n$, *i.e.* $E = L_1 \oplus L_2 \oplus \cdots \oplus L_n$. In what follows we will simplify notation by setting $c^i := c_1(L_i), i = 1, \ldots, n$. From axiom A2 we get that $c_t(E) = c_t(L_1) \cdot c_t(L_2) \cdot \cdots \cdot c_t(L_n) = (1 + c^1 t) \cdot (1 + c^2 t) \cdot \cdots \cdot (1 + c^n t)$. From this we conclude that

$$c_0(E) = 1,$$
$$c_1(E) = c^1 + c^2 + \cdots + c^n = \sum_{1 \leq i \leq n} c^i,$$

$$\begin{aligned} c_2(E) &= c^1 \cdot c^2 + \cdots + c^1 \cdot c^n + c^2 \cdot c^3 + c^2 \cdot c^4 + \cdots + c^2 \cdot c^n \\ &\quad + \cdots + c^{n-1} \cdot c^n \\ &= \sum_{1 \leq i < j \leq n} c^i \cdot c^j, \end{aligned}$$

$$c_3(E) = \sum_{1 \leq i < j < k \leq n} c^i \cdot c^j \cdot c^k,$$

$$\vdots$$

$$c_n(E) = \sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq n} c^{i_1} \cdot c^{i_2} \cdot \cdots \cdot c^{i_n}.$$

From these computations we realize that the different Chern classes of this vector bundle of rank n can just be obtained from the knowledge of the first Chern classes associated to the involved line bundles in the direct sum. Indeed, by axiom A2 we have just established that the total Chern class of E is given by $c(E) = (1 + c^1) \cdot (1 + c^2) \cdot \cdots \cdot (1 + c^n)$.
It is also interesting to observe here that the computations of the different Chern classes give rise to elementary symmetric polynomials in n variables $c^1, c^2, \ldots, c^n$.

Remark 2.4.10. Combining the results of the exercises 2.4.7 and 2.4.9 one immediately obtains that the total Chern class of the trivial bundle I of rank n which can be decomposed into the following direct sum $I = I_1 \oplus I_2 \oplus \cdots \oplus I_n$, where I_i is the trivial line bundle on X, is just $c(I) = c_0(I) = 1$, since $c_i(I) = 0$, for all $i = 1, \ldots, n$.

Example 2.4.11 (Application of axiom A1)**.** Let X be the complex projective space of dimension 2 ($X = \mathbb{CP}^2$). Consider $L := \{\langle x_1, x_2, 0\rangle \mid x_1, x_2 \in \mathbb{C}\}$. Clearly, we have that $L \subset \mathbb{CP}^2$ and moreover $L \simeq \mathbb{CP}^1$. Now, let E be a vector bundle of rank r on $\mathbb{CP}^2$ such that $E|_L \simeq L \times \mathbb{C}^r$. We are interested in computing the first Chern class of E.
Therefore, let us denote by i the inclusion map $L \overset{i}{\hookrightarrow} \mathbb{CP}^2$. We then get a map $H^*\left(\mathbb{CP}^2\right) \xrightarrow{i^*} H^*(L)$. Since $L \simeq \mathbb{CP}^1$ and $c_1(E) \in H^2(\mathbb{CP}^2, \mathbb{Z})$, we have that $i^*(c_1(E)) \in H^2(\mathbb{CP}^1, \mathbb{Z})$. Moreover, using Theorem 1.4.1 we have that $i^* \in \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \simeq \mathbb{Z}$. The map i^* is then injective since it is nontrivial (have a look at the next claim). By axiom A1 we have that $i^*(c_1(E)) = c_1(i^*E) = c_1(L \times \mathbb{C}^r) = 0$ since $L \times \mathbb{C}^r$ is a trivial bundle (cf. Remark 2.4.10). From the injectivity of i^* we conclude that $c_1(E) = 0$.
In fact, the map i^* is even an isomorphism, as it maps the canonical generator x of the cohomology group $H^2(\mathbb{CP}^2, \mathbb{Z})$ (*i.e.* $x = c_1(\mathscr{S}^*)$, where $\mathscr{S}$ is the tautological line bundle on $\mathbb{CP}^2$) to the canonical generator x' of the cohomology group $H^2(\mathbb{CP}^1, \mathbb{Z})$. Below we will prove this fact for the general case, *i.e.* for the inclusion map $i : \mathbb{CP}^1 \to \mathbb{CP}^n$.

Claim. $i^* : H^2(\mathbb{CP}^n, \mathbb{Z}) \to H^2(\mathbb{CP}^1, \mathbb{Z})$ *is an isomorphism.*

Proof. Recall that the pullback bundle π^*E of a vector bundle E of rank r generates the following diagram:

$$\begin{array}{ccccc} \pi^*E = \mathbb{P}(E) \underset{X}{\times} E \subset & \mathbb{P}(E) \times E & \longrightarrow & E \\ & \downarrow & & \downarrow p \\ & \mathbb{P}(E) & \xrightarrow[\pi]{} & X, \end{array}$$

and satisfies the equality $\pi(x) = p(v)$ for any tuple $(x, v) \in \pi^*E$. Now, consider the vector bundle $\mathbb{C}^{n+1} \to \{x\}$, *i.e.* the underlying manifold consists of one single point. Applying the *Leray-Hirsch* theorem to this vector bundle, we get that $-c_1\left(\mathcal{O}_{\mathbb{CP}^n}(-1)\right)$ generates $H^2(\mathbb{CP}^n, \mathbb{Z})$.
For $E = \mathbb{C}^{n+1}$ we get the following diagram:

$$\begin{array}{ccccc} \pi^*\mathbb{C}^{n+1} \subset & \mathbb{CP}^n \times \mathbb{C}^{n+1} & \longrightarrow & \mathbb{C}^{n+1} \\ & \downarrow & & \downarrow p \\ & \mathbb{CP}^n & \xrightarrow[\pi]{} & \{x\}. \end{array}$$

Moreover, notice that $i^*\left(\mathcal{O}_{\mathbb{CP}^n}(-1)\right) \simeq \mathcal{O}_{\mathbb{CP}^1}(-1)$.
Indeed: By definition of the tautological bundle $\mathscr{S}$ we have that $\mathscr{S} \subset \mathbb{CP}^n \times \mathbb{C}^{n+1}$. From this it follows that $i^*\mathscr{S} \subset i^*\left(\mathbb{CP}^n \times \mathbb{C}^{n+1}\right)$.

$$\begin{array}{ccccc} i^*\mathscr{S} \subset & i^*\left(\mathbb{CP}^n \times \mathbb{C}^{n+1}\right) & \longrightarrow & \mathbb{CP}^n \times \mathbb{C}^{n+1} & \supset \mathscr{S} \\ & \downarrow & & \downarrow p & \\ & \mathbb{CP}^1 & \xrightarrow{\quad i \quad} & \mathbb{CP}^n. & \end{array}$$

Let us have a closer look at $i^*\left(\mathbb{CP}^n \times \mathbb{C}^{n+1}\right)$. We have the following diagram:

$$\begin{array}{ccccc} i^*\left(\mathbb{CP}^n \times \mathbb{C}^{n+1}\right) \subset & \mathbb{CP}^1 \times \mathbb{CP}^n \times \mathbb{C}^{n+1} & \longrightarrow & \mathbb{CP}^n \times \mathbb{C}^{n+1} & \\ & \downarrow & & \downarrow p & \\ & \mathbb{CP}^1 & \xrightarrow[i]{\qquad\qquad} & \mathbb{CP}^n & \end{array}$$

and by definition

$$i^*\left(\mathbb{CP}^n \times \mathbb{C}^{n+1}\right) = \left\{ (x, y, v) \in \mathbb{CP}^1 \times \mathbb{CP}^n \times \mathbb{C}^{n+1} \,\middle|\, i(x) = p(y, v) \right\},$$

where $x := \langle x_0, x_1 \rangle$, $y := \langle y_0, y_1, \ldots, y_n \rangle$ and $v := (v_0, v_1, \ldots, v_n)$.
Then,

$$\begin{aligned} i(x) = p(y, v) &\Leftrightarrow i\left(\langle x_0, x_1 \rangle\right) = \langle y_0, y_1, \ldots, y_n \rangle \\ &\Leftrightarrow \langle x_0, x_1, 0. \ldots, 0 \rangle = \langle y_0, y_1, \ldots, y_n \rangle. \end{aligned}$$

This allows us to conclude that

$$\begin{aligned} i^*\left(\mathbb{CP}^n \times \mathbb{C}^{n+1}\right) &= \{(\langle x_0, x_1 \rangle, \langle x_0, x_1, 0, \ldots, 0 \rangle, (v_0, v_1, \ldots, v_n))\} \\ &\simeq \mathbb{CP}^1 \times \mathbb{C}^{n+1}, \end{aligned}$$

so that $i^*\mathscr{S}$ corresponds to the tautological subbundle in $\mathbb{CP}^1 \times \mathbb{C}^2 \subset \mathbb{CP}^1 \times \mathbb{C}^{n+1}$, *i.e.* $i^*\left(\mathcal{O}_{\mathbb{CP}^n}(-1)\right) \simeq \mathcal{O}_{\mathbb{CP}^1}(-1)$.
Effectively, we have that $i^*\mathscr{S} \subset \mathbb{CP}^1 \times \mathscr{S}$ and by definition we know that

$$\begin{aligned} & i^*\mathscr{S} \\ &= \left\{ (x, y, v) \in \mathbb{CP}^1 \times \mathscr{S} \,\middle|\, v \in y, i(x) = p(y, v) \right\} \\ &\simeq \{ (\langle x_0, x_1 \rangle, \langle x_0, x_1, 0, \ldots, 0 \rangle, (\lambda x_0, \lambda x_1, 0, \ldots, 0)) \mid \langle x_0, x_1 \rangle \in \mathbb{CP}^1, \lambda \in \mathbb{C} \} \\ &\simeq \{ (\langle x_0, x_1 \rangle, (\lambda x_0, \lambda x_1)) \mid \langle x_0, x_1 \rangle \in \mathbb{CP}^1, \lambda \in \mathbb{C} \}. \end{aligned}$$

Finally, as c_1 is a morphism of rings and using the fact that $i^*\left(\mathcal{O}_{\mathbb{CP}^n}(-1)\right) \simeq \mathcal{O}_{\mathbb{CP}^1}(-1)$ we conclude that

$$i^*\left(\underbrace{-c_1\left(\mathcal{O}_{\mathbb{CP}^n}(-1)\right)}_{\text{generator of } H^2(\mathbb{CP}^n,\mathbb{Z})}\right) = -c_1\left(i^*\left(\mathcal{O}_{\mathbb{CP}^n}(-1)\right)\right) = \underbrace{-c_1\left(\mathcal{O}_{\mathbb{CP}^1}(-1)\right)}_{\text{generator of } H^2(\mathbb{CP}^1,\mathbb{Z})}$$

Hence, $i^* : H^2(\mathbb{CP}^n,\mathbb{Z}) \to H^2(\mathbb{CP}^1,\mathbb{Z})$ is an isomorphism as it maps a generator to a generator. □

2.5 Chern Character & Todd Class

2.5.1 Chern Character

In order to give the definitions of Chern character and Todd class we need to understand the splitting principle which is a technique that enables questions about Chern classes of vector bundles to be reduced to the case of line bundles:

Theorem 2.5.1 (**The splitting principle**). *Let E be a vector bundle of rank r on a complex manifold X. Then there exists a holomorphic map of complex manifolds $f : Y \to X$ such that $H^*(X) \xrightarrow{f^*} H^*(Y)$ is injective and such that $c_i(f^*E) = f^*c_i(E)$. Moreover, there exist line bundles $L_1, \ldots, L_n$ on Y such that $c_t(f^*E) = \prod_{i=1}^r c_t(L_i)$.*

Proof. First, note that for a given holomorphic map of complex manifolds $f : Y \to X$ the formula $c_i(f^*E) = f^*c_i(E)$ is just axiom A1 of Grothendieck's axiomatic approach.
The result $c_t(\pi^*E) = \prod_{i=1}^r c_t(L_i)$ is obtained by induction on the rank of the vector bundle E. For the case of line bundles there is nothing to prove. For $r > 1$, we get a line subbundle $\mathscr{S}$ of π^*E on $\mathbb{P}(E)$ and therefore a short exact sequence of vector bundles on $\mathbb{P}(E)$:

$$0 \to \mathscr{S} \to \pi^*(E) \to Q \to 0.$$

where Q is the quotient bundle $\pi^*(E)/\mathscr{S}$. This exact sequence together with axiom A2 of Grothendieck's axiomatic approach allow us to derive by induction the desired formula $c_t(\pi^*E) = \prod_{i=1}^r c_t(L_i)$ for an arbitrary vector bundle E of rank r. Below, we will write down the computations for the cases $r = 2$ and $r = 3$ before proving the general case by induction.

(i) $\underline{r = 2}$
We take $Y = \mathbb{P}(E)$ and $f = \pi : \mathbb{P}(E) \to X$. Then by the *Leray-Hirsch* Theorem $H^*(X)$ can be naturally seen as a subring of $H^*(\mathbb{P}(E))$ via the homomorphism $\pi^* : H^*(X) \to H^*(\mathbb{P}(E))$. We know the following short exact sequence:

$$0 \to \underbrace{\mathscr{S}}_{1} \to \underbrace{\pi^*E}_{r} \to \underbrace{Q = \pi^*E/\mathscr{S}}_{1} \to 0,$$

where the numbers below the vector bundles indicate their rank.
By setting $L_1 := \mathscr{S}$ and $L_2 := Q$, we get the following short exact sequence of vector bundles on $\mathbb{P}(E)$:

$$0 \to L_1 \to \pi^*E \to L_2 \to 0,$$

which allows us to conclude by means of axiom A2 that

$$c_t(\pi^*E) = c_t(L_1) \cdot c_t(L_2).$$

(ii) $\underline{r = 3}$
We know the following short exact sequence:

(1) $$0 \to \underbrace{\mathscr{S}_1}_{1} \to \underbrace{\pi^*E}_{r} \to \underbrace{Q_1 = \pi^*E/\mathscr{S}_1}_{2} \to 0,$$

where S_1 denotes the line subbundle of π^*E on $\mathbb{P}(E)$.
We set $L_1 := \mathscr{S}_1$.
Now, consider the corresponding projective bundle $\mathbb{P}(Q_1) \xrightarrow{\pi_1} \mathbb{P}(E)$ and pullback Q_1 to $\mathbb{P}(Q_1)$ along π_1 in order to obtain $\pi_1^*Q_1$. Let us denote by S_2 the line subbundle of $\pi_1^*Q_1$ on $\mathbb{P}(Q_1)$. We then get the following short exact sequence:

$$0 \to \underbrace{\mathscr{S}_2}_{1} \to \pi_1^*Q_1 \to \underbrace{Q_2 = \pi_1^*Q_1/\mathscr{S}_2}_{1} \to 0.$$

We set $L_2 := \mathscr{S}_2$ and $L_3 := Q_2$. Since the pullback of a short exact sequence of vector bundles preserves the exactness of it, we can pullback the short exact sequence (1) along π_1 in order to get the following short exact sequences of vector bundles on $\mathbb{P}(Q_1)$:

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & & & & & L_2 & & \\ & & & & & & \downarrow & & \\ 0 & \to & \pi_1^* L_1 & \to & \pi_1^*\pi^* E & \to & \pi_1^* Q_1 & \to & 0 \\ & & & & & & \downarrow & & \\ & & & & & & L_3 & & \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array},$$

which allow us to conclude by means of axiom A2 that

$$c_t(\pi_1^*\pi^* E) = c_t(\pi_1^* L_1) \cdot c_t(L_2) \cdot c_t(L_3).$$

This gives us the desired result since through the *Leray-Hirsch* Theorem we have the following inclusion of modules: $H^*(X) \subset H^*(\mathbb{P}(E)) \subset H^*(\mathbb{P}(Q_1))$ so that finally: $(Y = \mathbb{P}(Q_1),\ f = \pi \circ \pi_1)$

$$c_t((\pi \circ \pi_1)^* E) = c_t(L_1') \cdot c_t(L_2) \cdot c_t(L_3),$$

where $L_1' := \pi_1^* L_1$.

(iii) $\underline{n \to n+1}$
Assume the Theorem is true for any vector bundle E of rank n on a complex manifold X, *i.e.*

$$c_t(f^* E) = \prod_{i=1}^{n} c_t(L_i),$$

where $f : Y \to X$ is holomorphic, $f^* : H^*(X) \to H^*(Y)$ is injective and $L_1, \ldots, L_n$ are line bundles on Y.
Let E' be a vector bundle of rank $n+1$ on X. We have the following short exact sequence of vector bundles on $\mathbb{P}(E')$:

$$0 \to \underbrace{\mathscr{S}}_{1} \to \underbrace{\pi^* E'}_{n+1} \to \underbrace{Q = \pi^* E'/\mathscr{S}}_{n} \to 0,$$

which allows us to conclude by means of axiom A2 that

$$c_t(\pi^* E') = c_t(\mathscr{S}) \cdot c_t(Q),$$

where Q is a vector bundle of rank n on $\mathbb{P}(E')$. Hence, we can apply our induction hypothesis in order to get a holomorphic map $f' : Y' \to \mathbb{P}(E')$ such that $c_t(f'^* Q) = \prod_{i=1}^{n} c_t(L_i')$, where $f'^* : H^*(\mathbb{P}(E')) \to$

$H^*(Y')$ is injective and $L'_1, \ldots, L'_n$ are line bundles on Y'. Now pullback $\mathscr{S}$ to Y' along f' respectively π^*E' to Y' along f' in order to get $f'^*\mathscr{S}$ respectively $f'^*\pi^*E'$ so that finally:

$$\begin{aligned} c_t((\pi \circ f')^*E') &= c_t(f'^*\mathscr{S}) \cdot c_t(f'^*Q) \\ &= c_t(f'^*\mathscr{S}) \prod_{i=1}^{n} c_t(L'_i) \\ &= \prod_{i=1}^{n+1} c_t(L'_i), \end{aligned}$$

where $L'_{n+1} := f'^*\mathscr{S}$. The map $\pi \circ f'$ is clearly holomorphic as it is the composition of 2 holomorphic maps and the injectivity of $(\pi \circ f')^*$ is also evident as it is the composition of 2 injective maps.
Finally, the injectivity of $f : Y \to X$ follows from the *Leray-Hirsch* Theorem by the construction of f.

□

Remark 2.5.2. The splitting principle allows us to conclude that if some formula with $c_i(\pi^*E)$ is true in $H^*(\mathbb{P}(E))$ then the "same" is valid for $c_i(E)$ in $H^*(X)$. This follows from the injectivity of the map f^*, which tells us that one can embed $H^*(X)$ in $H^*(\mathbb{P}(E))$. In other words, the Chern polynomial of a given vector bundle E of rank r can always be seen as $c_t(E) = \prod_{i=1}^{r}(1+a_i t)$ where the a_i's are just formal symbols called **Chern roots**. More precisely, one formally writes E as a direct sum of r line bundles, *i.e.* $E = \oplus_{i=1}^{r} L_i$ and one has that $a_i = c_1(L_i)$.

Exercise 2.5.3. (Based on Proposition 3.2. in [1].) Using the splitting principle, we can also calculate the Chern classes of tensor products easily. Let E be a vector bundle of rank r and E' a vector bundle of rank s. According to the splitting principle one can write E and E' formally as $E = L_1 \oplus \cdots \oplus L_r$ and $E' = L'_1 \oplus \cdots \oplus L'_s$ so that

$$c_t(E) = \prod_{i=1}^{r}(1 + a_i t)$$

and

$$c_t(E') = \prod_{i=1}^{s}(1 + b_i t),$$

where $a_1, \ldots, a_r$, $b_1, \ldots, b_s$ are just formal symbols such that $a_i = c_1(L_i)$ and

$b_j = c_1(L'_j)$. Then we have:

$$\begin{aligned} c(E \otimes E') &= c((L_1 \oplus \cdots \oplus L_r) \otimes (L'_1 \oplus \cdots \oplus L'_s)) \\ &= c\left(\sum_{i,j} L_i \otimes L'_j\right) \\ &= \prod_{i,j} c(L_i \otimes L'_j) \\ &= \prod_{i,j} (1 + c_1(L_i) + c_1(L'_j)) \\ &= \prod_{i,j} (1 + a_i + b_j), \end{aligned}$$

where we use the fact that $c_1(L_i \otimes L_j) = c_1(L_i) + c_1(L_j)$, (cf. Remark 2.3.3). Finally, we just showed that

$$c_t(E \otimes E') = \prod_{i,j} (1 + (a_i + b_j)t).$$

Definition 2.5.4 (**Chern Character**)**.** Let

$$c_t(E) = \prod_{i=1}^{r} (1 + a_i t),$$

where the a_i's are the Chern roots of E. Then we define the **Chern character** of E by

$$\mathrm{ch}(E) = \sum_{i=1}^{r} e^{a_i},$$

where

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots,$$

Remark 2.5.5. The Chern character is a symmetric expression in the a_i's, so by applying the splitting principle it can be expressed as a polynomial in the $c_i(E)$, with rational coefficients. Below, we give the first five terms of this polynomial. The computations are detailed in Exercise 2.5.7.

$$\begin{aligned} \mathrm{ch}(E) = r + c_1 + \frac{1}{2}\left(c_1^2 - 2c_2\right) + \frac{1}{6}\left(c_1^3 - 3c_1c_2 + 3c_3\right) \\ + \frac{1}{24}\left(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4\right) + \cdots \end{aligned}$$

where we set $c_i = c_i(E)$, $c_i = 0$ if $i > r$.
From this interpretation one easily concludes that $\mathrm{ch}(E) \in H^*(X, \mathbb{Z}) \otimes \mathbb{Q}$.

Example 2.5.6. Let us check the equality of the preceding Remark for some vector bundle E of rank 2.
Therefore, imagine for example that $E = L_1 \oplus L_2$, where L_1 and L_2 are two line bundles on X. In Example 2.4.8 we have established that $c(E) = \prod_{i=1}^{2}(1+c_1(L_i)) = 1+c_1(L_1)+c_1(L_2)+c_1(L_1)\cdot c_1(L_2)$. According to our formalism we have that $c_1(L_1) = a_1$ and $c_1(L_2) = a_2$ so that from the preceding equality we immediately conclude that $c_1(E) = a_1 + a_2$ and $c_2(E) = a_1 \cdot a_2$. By definition we have that

$$\begin{aligned}
\mathrm{ch}(E) &= \sum_{i=1}^{2} e^{a_i} \\
&= e^{a_1} + e^{a_2} \\
&= \left(1 + a_1 + \frac{1}{2}a_1^2 + \frac{1}{6}a_1^3 + \cdots\right) + \left(1 + a_2 + \frac{1}{2}a_2^2 + \frac{1}{6}a_2^3 + \cdots\right) \\
&= 2 + a_1 + a_2 + \frac{1}{2}\left(a_1^2 + a_2^2\right) + \frac{1}{6}\left(a_1^3 + a_2^3\right) + \cdots .
\end{aligned}$$

Now, we already know that $a_1 + a_2 = c_1(E)$. Moreover, we have that

$$\left(a_1^2 + a_2^2\right) = (a_1 + a_2)^2 - 2a_1a_2 = c_1(E)^2 - 2c_2(E)$$

and

$$\begin{aligned}
\left(a_1^3 + a_2^3\right) &= \left((a_1 + a_2)^3 - 3a_1^2a_2 - 3a_1a_2^2\right) \\
&= \left((a_1 + a_2)^3 - 3a_1a_2(a_1 + a_2)\right) \\
&= \left(c_1(E)^3 - 3c_2(E)c_1(E)\right).
\end{aligned}$$

This allows us to conclude that

$$\begin{aligned}
\mathrm{ch}(E) &= 2 + a_1 + a_2 + \frac{1}{2}\left(a_1^2 + a_2^2\right) + \frac{1}{6}\left(a_1^3 + a_2^3\right) + \cdots \\
&= 2 + c_1(E) + \frac{1}{2}\left(c_1(E)^2 - 2c_2(E)\right) \\
&\quad + \frac{1}{6}\left(c_1(E)^3 - 3c_1(E)c_2(E)\right) + \cdots
\end{aligned}$$

i.e. we end up with the equality given in Remark 2.5.5.

Exercise 2.5.7. Let us check the equality given in Remark 2.5.5 for the general case, and let us even try to compute the next term of it. Therefore, let E be an arbitrary vector bundle of rank n and $a_1, \ldots, a_n$ be the corresponding Chern roots, *i.e.*

$$\begin{aligned} c_1 &= \sum_i a_i, \\ c_2 &= \sum_{i<j} a_i a_j, \\ c_3 &= \sum_{i<j<k} a_i a_j a_k, \end{aligned}$$

$$\vdots$$

$$c_n = \sum_{i_1 < i_2 < \cdots < i_n} a_{i_1} a_{i_2} \cdots a_{i_n}.$$

By definition, we have that

$$\begin{aligned} &\mathrm{ch}(E) \\ &= \sum_{i=1}^{n} e^{a_i} \\ &= \sum_{i=1}^{n} \left(1 + a_i + \frac{a_i^2}{2} + \frac{a_i^3}{6} + \frac{a_i^4}{24} + \frac{a_i^5}{120} + \cdots\right) \\ &= n + \left(\sum_i a_i + \frac{1}{2}\sum_i a_i^2 + \frac{1}{6}\sum_i a_i^3 + \frac{1}{24}\sum_i a_i^4 + \frac{1}{120}\sum_i a_i^5 + \cdots\right) \\ &= n + c_1 + \left(\frac{1}{2}\sum_i a_i^2 + \frac{1}{6}\sum_i a_i^3 + \frac{1}{24}\sum_i a_i^4 + \frac{1}{120}\sum_i a_i^5 + \cdots\right). \end{aligned}$$

- $\underline{\sum_i a_i^2}$

$$\begin{aligned} \sum_i a_i^2 &= \left(\sum_i a_i\right)^2 - 2\sum_{i<j} a_i a_j \\ &\boxed{= c_1^2 - 2c_2} \end{aligned}$$

- $\underline{\sum_i a_i^3}$

$$\begin{aligned}
\sum_i a_i^3 &= \left(\sum_i a_i\right)^3 - 3\sum_{i\neq j} a_i^2 a_j - 6\sum_{i<j<k} a_i a_j a_k \\
&= c_1^3 - 3\sum_{i<j} a_i a_j (a_i + a_j) - 6c_3 \\
&= c_1^3 - 3\sum_{i<j} a_i a_j \left(c_1 - \sum_{\mu\neq i,j} a_\mu\right) - 6c_3 \\
&= c_1^3 - 3c_1c_2 + 3\sum_{i<j} a_i a_j \sum_{\mu\neq i,j} a_\mu - 6c_3 \\
&= c_1^3 - 3c_1c_2 + 9\sum_{i<j<k} a_i a_j a_k - 6c_3 \\
&= c_1^3 - 3c_1c_2 + 9c_3 - 6c_3 \\
&\boxed{= c_1^3 - 3c_1c_2 + 3c_3}
\end{aligned}$$

- $\underline{\sum_i a_i^4}$

$$\begin{aligned}
\sum_i a_i^4 = \left(\sum_i a_i\right)^4 &- 4\sum_{i\neq j} a_i^3 a_j - 6\sum_{i<j} a_i^2 a_j^2 \\
&- 12\sum_{\substack{i\neq j,k \\ j<k}} a_i^2 a_j a_k - 24\sum_{i<j<k<l} a_i a_j a_k a_l
\end{aligned}$$

$$\Leftrightarrow \sum_i a_i^4 = c_1^4 - 4\underbrace{\sum_{i\neq j} a_i^3 a_j}_{(1)} - 6\underbrace{\sum_{i<j} a_i^2 a_j^2}_{(2)} - 12\underbrace{\sum_{\substack{i\neq j,k \\ j<k}} a_i^2 a_j a_k}_{(3)} - 24c_4$$

(3)

$$\begin{aligned}\sum_{\substack{i\neq j,k\\ j<k}} a_i^2 a_j a_k &= \sum_{i<j<k} a_i a_j a_k (a_i + a_j + a_k)\\ &= \sum_{i<j<k} a_i a_j a_k \left(c_1 - \sum_{\mu\neq i,j,k} a_\mu \right)\\ &= c_1 c_3 - \sum_{i<j<k} a_i a_j a_k \sum_{\mu\neq i,j,k} a_\mu\\ &= c_1 c_3 - 4 \sum_{i<j<k<l} a_i a_j a_k a_l\\ &= c_1 c_3 - 4c_4\end{aligned}$$

(1)

$$\begin{aligned}\sum_{i\neq j} a_i^3 a_j &= \sum_{i<j} a_i a_j (a_i^2 + a_j^2)\\ &= \sum_{i<j} a_i a_j \left(\sum_i a_i^2 - \sum_{\mu\neq i,j} a_\mu^2 \right)\\ &= \sum_{i<j} a_i a_j \left(c_1^2 - 2c_2 - \sum_{\mu\neq i,j} a_\mu^2 \right)\\ &= c_1^2 c_2 - 2c_2^2 - \sum_{i<j} a_i a_j \sum_{\mu\neq i,j} a_\mu^2\\ &= c_1^2 c_2 - 2c_2^2 - \sum_{\substack{i\neq j,k\\ j<k}} a_i^2 a_j a_k\\ &= c_1^2 c_2 - 2c_2^2 - c_1 c_3 + 4c_4\end{aligned}$$

(2)

$$\begin{aligned}\sum_{i<j} a_i^2 a_j^2 &= \left(\sum_{i<j} a_i a_j \right)^2 - 2 \sum_{\substack{i\neq j,k\\ j<k}} a_i^2 a_j a_k - 6 \sum_{i<j<k<l} a_i a_j a_k a_l\\ &= c_2^2 - 2(c_1 c_3 - 4c_4) - 6c_4\\ &= c_2^2 - 2c_1 c_3 + 2c_4\end{aligned}$$

Finally,

$$\sum_i a_i^4 = c_1^4 - 4(c_1^2c_2 - 2c_2^2 - c_1c_3 + 4c_4) - 6(c_2^2 - 2c_1c_3 + 2c_4) - 12(c_1c_3 - 4c_4) - 24c_4$$

$\Leftrightarrow$

$$\sum_i a_i^4 = c_1^4 - 4c_1^2c_2 + 8c_2^2 + 4c_1c_3 - 16c_4 - 6c_2^2 + 12c_1c_3 - 12c_4 - 12c_1c_3 + 48c_4 - 24c_4$$

$\Leftrightarrow$

$$\boxed{\sum_i a_i^4 = c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4}$$

- $\underline{\sum_i a_i^5}$

$$\sum_i a_i^5 = \left(\sum_i a_i\right)^5 - 5\sum_{i\neq j} a_i^4a_j - 20\sum_{\substack{i\neq j,k\\ j<k}} a_i^3a_ja_k - 10\sum_{i\neq j} a_i^3a_j^2 - 30\sum_{\substack{k\neq i,j\\ i<j}} a_i^2a_j^2a_k - 60\sum_{\substack{i\neq j,k,l\\ j<k<l}} a_i^2a_ja_ka_l - 120\sum_{i<j<k<l<m} a_ia_ja_ka_la_m$$

$\Leftrightarrow$

$$\sum_i a_i^5 = c_1^5 - 5\underbrace{\sum_{i\neq j} a_i^4a_j}_{(1')} - 20\underbrace{\sum_{\substack{i\neq j,k\\ j<k}} a_i^3a_ja_k}_{(2')} - 10\underbrace{\sum_{i\neq j} a_i^3a_j^2}_{(3')} - 30\underbrace{\sum_{\substack{k\neq i,j\\ i<j}} a_i^2a_j^2a_k}_{(4')} - 60\underbrace{\sum_{\substack{i\neq j,k,l\\ j<k<l}} a_i^2a_ja_ka_l}_{(5')} - 120c_5$$

(5')

$$\begin{aligned}\sum_{\substack{i\neq j,k,l\\ j<k<l}} a_i^2 a_j a_k a_l &= \sum_{i<j<k<l} a_i a_j a_k a_l (a_i + a_j + a_k + a_l)\\ &= \sum_{i<j<k<l} a_i a_j a_k a_l \left(c_1 - \sum_{\mu\neq i,j,k,l} a_\mu \right)\\ &= c_1 c_4 - \sum_{i<j<k<l} a_i a_j a_k a_l \sum_{\mu\neq i,j,k,l} a_\mu\\ &= c_1 c_4 - 5 \sum_{i<j<k<l<m} a_i a_j a_k a_l a_m\\ &= c_1 c_4 - 5 c_5\end{aligned}$$

(2')

$$\begin{aligned}\sum_{\substack{i\neq j,k\\ j<k}} a_i^3 a_j a_k &= \sum_{i<j<k} a_i a_j a_k (a_i^2 + a_j^2 + a_k^2)\\ &= \sum_{i<j<k} a_i a_j a_k \left(\sum_i a_i^2 - \sum_{\mu\neq i,j,k} a_\mu^2 \right)\\ &= \sum_{i<j<k} a_i a_j a_k \left(c_1^2 - 2c_2 - \sum_{\mu\neq i,j,k} a_\mu^2 \right)\\ &= c_3(c_1^2 - 2c_2) - \sum_{i<j<k} a_i a_j a_k \sum_{\mu\neq i,j,k} a_\mu^2\\ &= c_1^2 c_3 - 2c_2 c_3 - \sum_{\substack{i\neq j,k,l\\ j<k<l}} a_i^2 a_j a_k a_l\\ &= c_1^2 c_3 - 2c_2 c_3 - c_1 c_4 + 5c_5\end{aligned}$$

(4')

$$\sum_{\substack{k\neq i,j\\ i<j}} a_i^2 a_j^2 a_k$$
$$= \sum_{i<j<k} a_i a_j a_k (a_i a_j + a_i a_k + a_j a_k)$$
$$= \sum_{i<j<k} a_i a_j a_k \left(c_2 - \sum_{\substack{\alpha,\beta\neq i,j,k\\ \alpha<\beta}} a_\alpha a_\beta - \sum_{\substack{\mu\neq\alpha,\beta\\ \nu\neq i,j,k\\ \mu<\nu}} a_\mu a_\nu \right)$$
$$= c_2 c_3 - \left(\sum_{i<j<k} a_i a_j a_k \sum_{\substack{\alpha,\beta\neq i,j,k\\ \alpha<\beta}} a_\alpha a_\beta \right) - \left(\sum_{i<j<k} a_i a_j a_k \sum_{\substack{\mu\neq\alpha,\beta\\ \nu\neq i,j,k\\ \mu<\nu}} a_\mu a_\nu \right)$$
$$= c_2 c_3 - 10 \sum_{i<j<k<l<m} a_i a_j a_k a_l a_m - 3 \sum_{\substack{i\neq j,k,l\\ j<k<l}} a_i^2 a_j a_k a_l$$
$$= c_2 c_3 - 10 c_5 - 3 c_1 c_4 + 15 c_5$$
$$= c_2 c_3 - 3 c_1 c_4 + 5 c_5$$

(1')

$$\sum_{i\neq j} a_i^4 a_j = \sum_{i<j} a_i a_j (a_i^3 + a_j^3)$$
$$= \sum_{i<j} a_i a_j \left(\sum_i a_i^3 - \sum_{\mu\neq i,j} a_\mu^3 \right)$$
$$= \sum_{i<j} a_i a_j \left(c_1^3 - 3 c_1 c_2 + 3 c_3 - \sum_{\mu\neq i,j} a_\mu^3 \right)$$
$$= c_1^3 c_2 - 3 c_1 c_2^2 + 3 c_2 c_3 - \sum_{i<j} a_i a_j \sum_{\mu\neq i,j} a_\mu^3$$
$$= c_1^3 c_2 - 3 c_1 c_2^2 + 3 c_2 c_3 - \sum_{\substack{i\neq j,k\\ j<k}} a_i^3 a_j a_k$$
$$= c_1^3 c_2 - 3 c_1 c_2^2 + 3 c_2 c_3 - c_1^2 c_3 + 2 c_2 c_3 + c_1 c_4 - 5 c_5$$
$$= c_1^3 c_2 - 3 c_1 c_2^2 + 5 c_2 c_3 - c_1^2 c_3 + c_1 c_4 - 5 c_5$$

(3')

$$\begin{aligned}
\sum_{i\neq j} a_i^3 a_j^2 &= \sum_{i<j} a_i^2 a_j^2 (a_i + a_j) \\
&= \sum_{i<j} a_i^2 a_j^2 \left(c_1 - \sum_{\mu\neq i,j} a_u \right) \\
&\overset{(2)}{=} c_1(c_2^2 - 2c_1c_3 + 2c_4) - \sum_{i<j} a_i^2 a_j^2 \sum_{\mu\neq i,j} a_\mu \\
&= c_1c_2^2 - 2c_1^2c_3 + 2c_1c_4 - \sum_{\substack{k\neq i,j \\ i<j}} a_i^2 a_j^2 a_k \\
&= c_1c_2^2 - 2c_1^2c_3 + 2c_1c_4 - c_2c_3 + 3c_1c_4 - 5c_5 \\
&= c_1c_2^2 - 2c_1^2c_3 + 5c_1c_4 - c_2c_3 - 5c_5
\end{aligned}$$

Finally,

$$\begin{aligned}
\sum_i a_i^5 = c_1^5 &- 5(c_1^3c_2 - 3c_1c_2^2 + 5c_2c_3 - c_1^2c_3 + c_1c_4 - 5c_5) \\
&- 20(c_1^2c_3 - 2c_2c_3 - c_1c_4 + 5c_5) \\
&- 10(c_1c_2^2 - 2c_1^2c_3 + 5c_1c_4 - c_2c_3 - 5c_5) \\
&- 30(c_2c_3 - 3c_1c_4 + 5c_5) - 60(c_1c_4 - 5c_5) - 120c_5
\end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned}
\sum_i a_i^5 = c_1^5 &- 5c_1^3c_2 + 15c_1c_2^2 - 25c_2c_3 + 5c_1^2c_3 - 5c_1c_4 + 25c_5 \\
&- 20c_1^2c_3 + 40c_2c_3 + 20c_1c_4 - 100c_5 - 10c_1c_2^2 + 20c_1^2c_3 \\
&- 50c_1c_4 + 10c_2c_3 + 50c_5 - 30c_2c_3 + 90c_1c_4 - 150c_5 \\
&- 60c_1c_4 + 300c_5 - 120c_5
\end{aligned}$$

$\Leftrightarrow$

$$\boxed{\sum_i a_i^5 = c_1^5 - 5c_1^3c_2 + 5c_1c_2^2 + 5c_1^2c_3 - 5c_2c_3 - 5c_1c_4 + 5c_5}$$

It turns out that there exists a nice recursive formula generating each term of the Chern character of a vector bundle in terms of its Chern classes.

Let E be a vector bundle of rank n and let $a_1, \ldots, a_n$ be its Chern roots. Define

$$\Sigma_r := \sum_{i=1}^{n} a_i^r,$$

i.e. the degree r term ch_r of the Chern character $\mathrm{ch}(E)$ is given by $\mathrm{ch}_r := \frac{1}{r!}\Sigma_r$. We then have the following Proposition:

Proposition 2.5.8. *Let E be a vector bundle of rank n and let $a_1, \ldots, a_n$ be its Chern roots. Then,*

$$\boxed{\Sigma_{r+1} = c_1\Sigma_r - c_2\Sigma_{r-1} + c_3\Sigma_{r-2} - \cdots + (-1)^r(r+1)c_{r+1}.}$$

Proof. For $r = 0$, we get

$$\Sigma_1 = c_1.$$

For $r = 1$, we get

$$\begin{aligned} \Sigma_2 &= c_1\Sigma_1 - 2c_2 \\ &= c_1^2 - 2c_2. \end{aligned}$$

For $r \geq 2$, we have that

$$\begin{aligned} \Sigma_r \cdot \Sigma_1 &= \Sigma_r \cdot c_1 \\ &= \left(\sum_i a_i^r\right)\left(\sum_i a_i\right) \\ &= \sum_i a_i^{r+1} + \sum_{i_1 \neq i_2} a_{i_1}^r a_{i_2} \\ &= \Sigma_{r+1} + \sum_{i_1 \neq i_2} a_{i_1}^r a_{i_2}, \end{aligned}$$

i.e.

$$\Sigma_{r+1} = c_1\Sigma_r - \sum_{i_1 \neq i_2} a_{i_1}^r a_{i_2}.$$

Now,

$$\begin{aligned} \sum_{i_1 \neq i_2} a_{i_1}^r a_{i_2} &= \sum_{i_1 < i_2} a_{i_1} a_{i_2}(a_{i_1}^{r-1} + a_{i_2}^{r-1}) \\ &= \sum_{i_1 < i_2} a_{i_1} a_{i_2} \left(\Sigma_{r-1} - \sum_{\mu \neq i_1, i_2} a_\mu^{r-1}\right) \\ &= c_2 \cdot \Sigma_{r-1} - \sum_{i_1 < i_2} a_{i_i} a_{i_2} \sum_{\mu \neq i_1, i_2} a_\mu^{r-1}. \end{aligned}$$

Notice that,

$$\sum_{i_1<i_2<\cdots<i_p} a_{i_1}a_{i_2}\dots a_{i_p} \sum_{\mu\neq i_1,\dots,i_p} a_\mu^q$$

$$= \begin{cases} \sum_{i_1<i_2<\cdots<i_{p+1}} a_{i_1}a_{i_2}\dots a_{i_p}a_{i_{p+1}} \left(a_{i_1}^{q-1}+a_{i_2}^{q-1}+\cdots+a_{i_{p+1}}^{q-1}\right) & q\geq 2 \\ (p+1)\sum_{i_1<i_2<\cdots<i_{p+1}} a_{i_1}a_{i_2}\dots a_{i_p}a_{i_{p+1}} & q=1 \end{cases}$$

$$= \begin{cases} c_{p+1}\cdot\Sigma_{q-1} - \sum_{i_1<i_2<\cdots<i_{p+1}} a_{i_1}\dots a_{i_{p+1}} \sum_{\mu\neq i_1,\dots,i_{p+1}} a_\mu^{q-1} & q\geq 2 \\ (p+1)c_{p+1} & q=1. \end{cases}$$

Applying this recursively yields the desired formula

$$\boxed{\Sigma_{r+1} = c_1\Sigma_r - c_2\Sigma_{r-1} + c_3\Sigma_{r-2} - \cdots + (-1)^r(r+1)c_{r+1}.}$$

□

Example 2.5.9. Let us apply the recursive formula seen in Proposition 2.5.8 for $r = 2, 3, 4, 5$.

- $\underline{r=2}$

$$\begin{aligned} \Sigma_3 &= c_1\Sigma_2 - c_2\Sigma_1 + 3c_3 \\ &= c_1(c_1^2-2c_2) - c_1c_2 + 3c_3 \\ &= c_1^3 - 3c_1c_2 + 3c_3 \end{aligned}$$

- $\underline{r=3}$

$$\begin{aligned} \Sigma_4 &= c_1\Sigma_3 - c_2\Sigma_2 + c_3\Sigma_1 - 4c_4 \\ &= c_1(c_1^3 - 3c_1c_2 + 3c_3) - c_2(c_1^2-2c_2) + c_1c_3 - 4c_4 \\ &= c_1^4 - 3c_1^2c_2 + 3c_1c_3 - c_1^2c_2 + 2c_2^2 + c_1c_3 - 4c_4 \\ &= c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4 \end{aligned}$$

- $\underline{r=4}$

$$\begin{aligned} \Sigma_5 &= c_1\Sigma_4 - c_2\Sigma_3 + c_3\Sigma_2 - c_4\Sigma_1 + 5c_5 \\ &= c_1(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4) - c_2(c_1^3 - 3c_1c_2 + 3c_3) \\ &\quad + c_3(c_1^2 - 2c_2) - c_1c_4 + 5c_5 \\ &= c_1^5 - 4c_1^3c_2 + 4c_1^2c_3 + 2c_1c_2^2 - 4c_1c_4 - c_1^3c_2 + 3c_1c_2^2 - 3c_2c_3 \\ &\quad + c_1^2c_3 - 2c_2c_3 - c_1c_4 + 5c_5 \\ &= c_1^5 - 5c_1^3c_2 + 5c_1c_2^2 + 5c_1^2c_3 - 5c_2c_3 - 5c_1c_4 + 5c_5 \end{aligned}$$

All of this is in agreement with the computations we have already made in Exercise 2.5.7. Let us now compute with help of this recursive formula the 6^{th} term of the Chern character:

- $\underline{r = 5}$

$$\begin{aligned}
\Sigma_6 &= c_1\Sigma_5 - c_2\Sigma_4 + c_3\Sigma_3 - c_4\Sigma_2 + c_5\Sigma_1 - 6c_6 \\
&= c_1(c_1^5 - 5c_1^3c_2 + 5c_1c_2^2 + 5c_1^2c_3 - 5c_2c_3 - 5c_1c_4 + 5c_5) \\
&\quad - c_2(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4) + c_3(c_1^3 - 3c_1c_2 + 3c_3) \\
&\quad - c_4(c_1^2 - 2c_2) + c_1c_5 - 6c_6 \\
&= c_1^6 - 5c_1^4c_2 + 5c_1^2c_2^2 + 5c_1^3c_3 - 5c_1c_2c_3 - 5c_1^2c_4 + 5c_1c_5 \\
&\quad - c_1^4c_2 + 4c_1^2c_2^2 - 4c_1c_2c_3 - 2c_2^3 + 4c_2c_4 + c_1^3c_3 - 3c_1c_2c_3 + 3c_3^2 \\
&\quad - c_1^2c_4 + 2c_2c_4 + c_1c_5 - 6c_6 \\
&= c_1^6 - 6c_1^4c_2 + 6c_1^3c_3 + 9c_1^2c_2^2 - 12c_1c_2c_3 - 6c_1^2c_4 + 6c_1c_5 \\
&\quad + 6c_2c_4 + 3c_3^2 - 2c_2^3 - 6c_6
\end{aligned}$$

Remark 2.5.10. The expressions we have established here for Σ_r, are known under the name of Newton's identities.

Example 2.5.11. Let L_1 and L_2 be two line bundles on X. Then using the equality given in Remark 2.5.5, one concludes that

$$\mathrm{ch}(L_1) = 1 + c_1(L_1) + \frac{1}{2}c_1(L_1)^2 + \frac{1}{6}c_1(L_1)^3 + \cdots = e^{c_1(L_1)}$$

and

$$\mathrm{ch}(L_2) = 1 + c_1(L_2) + \frac{1}{2}c_1(L_2)^2 + \frac{1}{6}c_1(L_2)^3 + \cdots = e^{c_1(L_2)}.$$

Moreover, using the definition, we have already seen in Example 2.5.6 that

$$\begin{aligned}
\mathrm{ch}(L_1 \oplus L_2) = 2 + c_1(L_1) + c_1(L_2) + \frac{1}{2}\left(c_1(L_1)^2 + c_1(L_2)^2\right) \\
+ \frac{1}{6}\left(c_1(L_1)^3 + c_1(L_2)^3\right) + \cdots,
\end{aligned}$$

which is nothing else than the sum $\mathrm{ch}(L_1) + \mathrm{ch}(L_2)$.
One could also get this result directly from the definition:

$$\mathrm{ch}(L_1 \oplus L_2) = e^{c_1(L_1)} + e^{c_1(L_2)} = \mathrm{ch}(L_1) + \mathrm{ch}(L_2).$$

Furthermore, we have already seen in Remark 2.3.3 that $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$, so that using the equality given in Remark 2.5.5 we get that

$$\mathrm{ch}(L_1 \otimes L_2) = 1 + c_1(L_1) + c_1(L_2) + \frac{1}{2}\left(c_1(L_1) + c_1(L_2)\right)^2 + \frac{1}{6}\left(c_1(L_1) + c_1(L_2)\right)^3 + \cdots,$$

which is nothing else than the product $\mathrm{ch}(L_1) \cdot \mathrm{ch}(L_2)$.
One could also get this result directly from the definition using the properties of the exponential map:

$$\mathrm{ch}(L_1 \otimes L_2) = e^{c_1(L_1 \otimes L_2)} = e^{c_1(L_1)+c_1(L_2)} = e^{c_1(L_1)} e^{c_1(L_2)} = \mathrm{ch}(L_1) \cdot \mathrm{ch}(L_2).$$

For the next considerations around the Chern character, we need to introduce the **Grothendieck group** $K(X)$ in the context of vector bundles.

Definition 2.5.12 (**Grothendieck group**)**.** Let $V(X)$ denote the free abelian group generated by all vector bundles on X. Consider $I(X)$, the subgroup generated by $E'' - E + E'$ for vector bundles E, E', E'' which define a short exact sequence $0 \to E'' \to E \to E' \to 0$. Then the **Grothendieck group** $K(X)$ is defined as

$$K(X) := {}^{V(X)}\!/_{I(X)},$$

where the sum of $[E_1], [E_2] \in K(X)$ is defined by

$$[E_1] + [E_2] := [E_1 + E_2].$$

Remark 2.5.13.

(i) It follows immediately from the preceding definition that $[E_1 \oplus E_2] = [E_1] + [E_2]$. Effectively, we then have a short exact sequence $0 \to E_1 \to E_1 \oplus E_2 \to E_2 \to 0$, *i.e.* $E_1 - (E_1 \oplus E_2) + E_2 \in I(X)$. Hence,

$$\begin{aligned}[E_1 - (E_1 \oplus E_2) + E_2] = [0] &\Leftrightarrow [E_1] - [E_1 \oplus E_2] + [E_2] = [0] \\ &\Leftrightarrow [E_1 \oplus E_2] = [E_1] + [E_2].\end{aligned}$$

(ii) Let E_1, E_2 be two vector bundles on X. If $E_1 \simeq E_2$ as vector bundles, then there exists a short exact sequence $0 \to 0_{VB} \to E_1 \to E_2 \to 0$, where 0_{VB} denotes the identically 0-vector bundle on X, *i.e.* the fibers correspond all to the $\{0\}$-vector space. From this we conclude that $[E_1] = [E_2] + [0_{VB}]$. However, we can write $0_{VB} = 0_{VB} - 0_{VB} + 0_{VB}$ and we have that $0 \to 0_{VB} \to 0_{VB} \to 0_{VB} \to 0$ is a short exact sequence, so that $0_{VB} \in I(X)$. This means that $[0_{VB}] = [0]$ which allows us to conclude that $[E_1] = [E_2]$.

Proposition 2.5.14. *The tensor product endows $K(X)$ with a commutative ring structure. More precisely, for $[E_1], [E_2] \in K(X)$ the product in $K(X)$ is defined in the following way:*

$$[E_1] \cdot [E_2] := [E_1 \otimes E_2].$$

Proof. Let us check that this is well-defined and indeed induces a ring structure on $K(X)$.

- Well-defined
  Let E_1 and E_1' be two representatives of the same class as well as E_2 and E_2', *i.e.* $[E_1] = [E_1']$ and $[E_2] = [E_2']$. We want to show that $[E_1 \otimes E_2] = [E_1' \otimes E_2']$. In other words, we need to prove that $E_1 \otimes E_2 - E_1' \otimes E_2'$ is an element of $I(X)$. Since $[E_1] = [E_1']$, we know that $E_1 - E_1' = \sum_i (F_i'' - F_i + F_i')$ where $0 \to F_i'' \to F_i \to F_i' \to 0$ is a short exact sequence of vector bundles for all i. Since $[E_2] = [E_2']$, we know that $E_2 - E_2' = \sum_i (G_i'' - G_i + G_i')$ where $0 \to G_i'' \to G_i \to G_i' \to 0$ is a short exact sequence of vector bundles for all i.
  We can write the initial difference in the following way:

  $$E_1 \otimes E_2 - E_1' \otimes E_2' = E_1 \otimes E_2 - E_1 \otimes E_2' + E_1 \otimes E_2' - E_1' \otimes E_2'.$$

  Now, we have that

  $$E_1 \otimes E_2 - E_1 \otimes E_2' = \sum_i \left(E_1 \otimes G_i'' - E_1 \otimes G_i + E_1 \otimes G_i' \right),$$

  where $0 \to E_1 \otimes G_i'' \to E_1 \otimes G_i \to E_1 \otimes G_i' \to 0$ is a short exact sequence for all i, as tensorising by vector bundles does not affect the exactness of a given short exact sequence. This shows that $E_1 \otimes E_2 - E_1 \otimes E_2' \in I(X)$.
  We also have that

  $$E_1 \otimes E_2' - E_1' \otimes E_2' = \sum_i \left(F_i'' \otimes E_2' - F_i \otimes E_2' + F_i' \otimes E_2' \right),$$

  where $0 \to F_i'' \otimes E_2' \to F_i \otimes E_2' \to F_i' \otimes E_2' \to 0$ is a short exact sequence for all i, because again tensorising by vector bundles does not affect the exactness of a given short exact sequence. This shows that $E_1 \otimes E_2' - E_1' \otimes E_2' \in I(X)$. Since both summands are in $I(X)$, this allows us to conclude that $E_1 \otimes E_2 - E_1' \otimes E_2'$ is an element of $I(X)$.

- Ring structure

  1. Associativity:
     Let $[E_1], [E_2], [E_3] \in K(X)$. Then,

     $$([E_1] \cdot [E_2]) \cdot [E_3] \overset{\text{def}}{=} [E_1 \otimes E_2] \cdot [E_3] \overset{\text{def}}{=} [(E_1 \otimes E_2) \otimes E_3].$$

However, $(E_1 \otimes E_2) \otimes E_3 \simeq E_1 \otimes (E_2 \otimes E_3)$ as vector bundles. Using Remark 2.5.13 we finally get:

$$\begin{aligned} ([E_1] \cdot [E_2]) \cdot [E_3] &\overset{\text{def}}{=} [(E_1 \otimes E_2) \otimes E_3] \\ &\overset{\text{rem(ii)}}{=} [E_1 \otimes (E_2 \otimes E_3)] \\ &\overset{\text{def}}{=} [E_1] \cdot [E_2 \otimes E_3] \\ &\overset{\text{def}}{=} [E_1] \cdot ([E_2] \cdot [E_3]) \,. \end{aligned}$$

This shows associativity.

2. Distributivity:
Let $[E_1], [E_2], [E_3] \in K(X)$. Then,

$$([E_1] + [E_2]) \cdot [E_3] \overset{\text{rem(i)}}{=} [E_1 \oplus E_2] \cdot [E_3] \overset{\text{def}}{=} [(E_1 \oplus E_2) \otimes E_3].$$

However, $(E_1 \oplus E_2) \otimes E_3 \simeq E_1 \otimes E_3 \oplus E_2 \otimes E_3$ as vector bundles. Applying Remark 2.5.13 we finally get:

$$\begin{aligned} ([E_1] + [E_2]) \cdot [E_3] &\overset{\text{def+rem(i)}}{=} [(E_1 \oplus E_2) \otimes E_3] \\ &\overset{\text{rem(ii)}}{=} [E_1 \otimes E_3 \oplus E_2 \otimes E_3] \\ &\overset{\text{rem(i)}}{=} [E_1 \otimes E_3] + [E_2 \otimes E_3] \\ &\overset{\text{def}}{=} [E_1] \cdot [E_3] + [E_2] \cdot [E_3]. \end{aligned}$$

The equality $[E_1] \cdot ([E_2] + [E_3]) = [E_1] \cdot [E_2] + [E_1] \cdot [E_3]$ is obtained in a similar way.

3. Neutral element:
Let I denote the trivial line bundle on X. We know that for any vector bundle E on X we have that $I \otimes E \simeq E$ as vector bundles. Using Remark 2.5.13 we get that:

$$[I] \cdot [E] \overset{\text{def}}{=} [I \otimes E] \overset{\text{rem(ii)}}{=} [E].$$

The equality in the other way around will immediately follow from the commutativity of this ring structure which we will prove as a last step.

4. Commutativity:
   Let $[E_1], [E_2] \in K(X)$. We know that $E_1 \otimes E_2 \simeq E_2 \otimes E_1$ as vector bundles, so by applying Remark 2.5.13 we conclude that

$$[E_1] \cdot [E_2] \overset{\text{def}}{=} [E_1 \otimes E_2] \overset{\text{rem(ii)}}{=} [E_2 \otimes E_1] \overset{\text{def}}{=} [E_2] \cdot [E_1].$$

□

Remark 2.5.15. More details about $K(X)$ can be found in section 2.1. of [8]. In Example 2.5.11 we have established that $\mathrm{ch}(L_1 \oplus L_2) = \mathrm{ch}(L_1) + \mathrm{ch}(L_2)$ and that $\mathrm{ch}(L_1 \otimes L_2) = \mathrm{ch}(L_1) \cdot \mathrm{ch}(L_2)$ for any two line bundles L_1, L_2 on X. It turns out that this property holds true for any two classes of vector bundles $[E], [E']$ on X from the *Grothendieck group* $K(X)$. More precisely, the Chern character is a ring homomorphism $\mathrm{ch} : K(X) \to H^*(X, \mathbb{Z}) \otimes \mathbb{Q}$. This is exactly the content of the following Theorem:

Theorem 2.5.16. *The Chern character is a ring homomorphism*

$$\mathrm{ch} : K(X) \to H^*(X, \mathbb{Z}) \otimes \mathbb{Q}.$$

Proof. (Based on Proposition 28.4. in [14] and on Proposition 3.2. in [1].) Let $E \in V(X)$ be a vector bundle of rank r and $E' \in V(X)$ be a vector bundle of rank s. By the splitting principle we can formally write both vector bundles as a direct sum of line bundles, *i.e.* $E = \oplus_{i=1}^{r} L_i$ and $E' = \oplus_{j=1}^{s} L'_j$. By linearity, we get a map $\mathrm{ch} : V(X) \to H^*(X, \mathbb{Z}) \otimes \mathbb{Q}$ such that
(1)

$$\mathrm{ch}(E \oplus E') = \mathrm{ch}(\oplus_{i=1}^{r} L_i \oplus_{j=1}^{s} L'_j) = \sum_{i=1}^{r} e^{c_1(L_i)} + \sum_{j=1}^{s} e^{c_1(L'_j)} = \mathrm{ch}(E) + \mathrm{ch}(E').$$

This shows that $V(X) \xrightarrow{\mathrm{ch}} H^*(X, \mathbb{Z}) \otimes \mathbb{Q}$ is a homomorphism of abelian groups. In Remark 2.5.17 following this theorem we will see that for a given short exact sequence $0 \to E'' \to E \to E' \to 0$, we have that $\mathrm{ch}(E) = \mathrm{ch}(E'') + \mathrm{ch}(E')$. Together with the fact that $\mathrm{ch} : V(X) \to H^*(X, \mathbb{Z})$ is a homomorphism of abelian groups, we get the following equality for any vector bundles E, E', E'' which define a short exact sequence $0 \to E'' \to E \to E' \to 0$:

$$\begin{aligned} \mathrm{ch}(E'' - E + E') &= \mathrm{ch}(E'') - \mathrm{ch}(E) + \mathrm{ch}(E') \\ &= \mathrm{ch}(E'') - (\mathrm{ch}(E'') + \mathrm{ch}(E')) + \mathrm{ch}(E') \\ &= 0, \end{aligned}$$

which shows that $I(X) \subset \mathrm{Ker}\ \mathrm{ch}$. Through the universal property, this allows us to conclude that there exists a unique map

$$\mathrm{ch} : {}^{V(X)}\!/_{I(X)} = K(X) \to H^*(X, \mathbb{Z}) \otimes \mathbb{Q},$$

which is naturally defined in the following way:

$$\mathrm{ch}\left([E]\right) := \mathrm{ch}(E) \quad \forall E \in V(X).$$

Moreover, in Exercise 2.5.3 we have already seen that $c_t(E \otimes E') = \prod_{i,j}(1 + (a_i + b_j)t)$, so that

$$\mathrm{ch}(E \otimes E') = \sum_{i=1}^{r}\sum_{j=1}^{s} e^{a_i+b_j} = \sum_{i=1}^{r} e^{a_i} \cdot \sum_{j=1}^{s} e^{b_j} = \mathrm{ch}(E) \cdot \mathrm{ch}(E'). \tag{2}$$

Finally, equations (1) and (2) together with Remark 2.5.13 allow us to conclude that

$$\begin{aligned} \mathrm{ch}\left([E] + [E']\right) &\overset{\text{rem(i)}}{=} \mathrm{ch}\left([E \oplus E']\right) \\ &\overset{\text{def}}{=} \mathrm{ch}\left(E \oplus E'\right) \\ &\overset{(1)}{=} \mathrm{ch}(E) + \mathrm{ch}(E') \\ &\overset{\text{def}}{=} \mathrm{ch}\left([E]\right) + \mathrm{ch}\left([E']\right) \end{aligned}$$

and

$$\begin{aligned} \mathrm{ch}\left([E] \cdot [E']\right) &\overset{\text{def}}{=} \mathrm{ch}\left([E \otimes E']\right) \\ &\overset{\text{def}}{=} \mathrm{ch}\left(E \otimes E'\right) \\ &\overset{(2)}{=} \mathrm{ch}(E) \cdot \mathrm{ch}(E') \\ &\overset{\text{def}}{=} \mathrm{ch}\left([E]\right) \cdot \mathrm{ch}\left([E']\right), \end{aligned}$$

i.e. the Chern character $\mathrm{ch} : K(X) \to H^*(X, \mathbb{Z}) \otimes \mathbb{Q}$ is a ring homomorphism. □

Remark 2.5.17. In Example 2.4.8 we have seen that Chern classes do not distinguish different extensions. This property is naturally inherited by the Chern character as it is defined by means of Chern classes. More precisely, for a given short exact sequence $0 \to E'' \to E \to E' \to 0$ of vector bundles on X, we have that

$$\mathrm{ch}(E) = \mathrm{ch}(E'') + \mathrm{ch}(E').$$

Effectively, we already know through axiom A2 that $c(E) = c(E') \cdot c(E'')$. As Chern classes do not distinguish different extensions we also have through axiom A2 that $c(E' \oplus E'') = c(E') \cdot c(E'')$ (We considered here the short exact sequence $0 \to E' \to E' \oplus E'' \to E'' \to 0$). Combining these two equalities we get that $c(E) = c(E' \oplus E'')$, *i.e.* $c_i(E) = c_i(E' \oplus E'')$. This allows us to conclude that $\mathrm{ch}(E) = \mathrm{ch}(E' \oplus E'')$. However, through Theorem 2.5.16 we know that $\mathrm{ch}(E' \oplus E'') = \mathrm{ch}(E') + \mathrm{ch}(E'')$ so that finally, $\mathrm{ch}(E) = \mathrm{ch}(E'') + \mathrm{ch}(E')$.

Exercise 2.5.18. Let E be a vector bundle of rank r and E' a vector bundle of rank s. In Exercise 2.5.3 we have established a formula giving the total Chern class of $E \otimes E'$ in terms of the corresponding Chern roots. From the Chern character property $\mathrm{ch}(E \otimes E') = \mathrm{ch}(E) \cdot \mathrm{ch}(E')$ one can deduce a formula in terms of the corresponding Chern classes. Below, we will write down the computations for the first and second Chern class of $E \otimes E'$.
By definition, we know that

$$\begin{aligned} \mathrm{ch}(E) &= r + c_1(E) + \frac{1}{2}\left(c_1(E)^2 - 2c_2(E)\right) + \cdots, \\ \mathrm{ch}(E') &= s + c_1(E') + \frac{1}{2}\left(c_1(E')^2 - 2c_2(E')\right) + \cdots. \end{aligned}$$

On the one hand, we have that

$$\mathrm{ch}(E \otimes E') = rs + c_1(E \otimes E') + \frac{1}{2}\left(c_1(E \otimes E')^2 - 2c_2(E \otimes E')\right) + \cdots.$$

On the other hand, we know that

$$\begin{aligned} \mathrm{ch}(E \otimes E') &= \mathrm{ch}(E) \cdot \mathrm{ch}(E') \\ &= rs + rc_1(E') + sc_1(E) + c_1(E)c_1(E') \\ &\quad + \frac{r}{2}\left(c_1(E')^2 - 2c_2(E')\right) + \frac{s}{2}\left(c_1(E)^2 - 2c_2(E)\right) + \cdots. \end{aligned}$$

By identification of the preceding equations one concludes that

$$\boxed{c_1(E \otimes E') = rc_1(E') + sc_1(E),}$$

$$\begin{aligned} &\frac{1}{2}\left(c_1(E \otimes E')^2 - 2c_2(E \otimes E')\right) \\ &= c_1(E)c_1(E') + \frac{r}{2}\left(c_1(E')^2 - 2c_2(E')\right) + \frac{s}{2}\left(c_1(E)^2 - 2c_2(E)\right) \end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned} c_2(E \otimes E') &= \frac{1}{2}c_1(E \otimes E')^2 \\ &\quad - \left(c_1(E)c_1(E') + \frac{r}{2}(c_1(E')^2 - 2c_2(E')) + \frac{s}{2}(c_1(E)^2 - 2c_2(E))\right) \end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned} c_2(E \otimes E') &= \frac{1}{2}\left(rc_1(E') + sc_1(E)\right)^2 - \left(\frac{r}{2}c_1(E')^2 + \frac{s}{2}c_1(E)^2 + c_1(E)c_1(E')\right) \\ &\quad + rc_2(E') + sc_2(E) \end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned} c_2(E\otimes E') = &\frac{1}{2}\left(r^2(c_1(E')^2 + s^2(c_1(E))^2 + 2rsc_1(E)c_1(E')\right) \\ &- \frac{1}{2}\left(rc_1(E')^2 + sc_1(E)^2 + 2c_1(E)c_1(E')\right) \\ &+ rc_2(E') + sc_2(E) \end{aligned}$$

$\Leftrightarrow$

$$\boxed{\begin{aligned} c_2(E\otimes E') = &\frac{1}{2}(r^2-r)c_1(E')^2 + \frac{1}{2}(s^2-s)(c_1(E))^2 \\ &+ (rs-1)c_1(E)c_1(E') + rc_2(E') + sc_2(E). \end{aligned}}$$

Example 2.5.19. Let us check the formulas we have discovered in the preceding Exercise for E and E' being vector bundles of rank 2, *i.e.* $r = 2$ and $s = 2$. We expect that

$$\begin{aligned} c_1(E\otimes E') &= 2c_1(E') + 2c_1(E), \\ c_2(E\otimes E') &= c_1(E')^2 + c_1(E)^2 + 3c_1(E)c_1(E') + 2c_2(E') + 2c_2(E). \end{aligned}$$

According to the splitting principle one can formally write E and E' as direct sum of line bundles

$$E = L_1 \oplus L_2, \qquad\qquad E' = L_1' \oplus L_2',$$

and express their Chern classes in terms of their Chern roots

$$\begin{aligned} c_1(E) &= a_1 + a_2, & c_2(E) &= a_1a_2, \\ c_1(E') &= b_1 + b_2, & c_2(E') &= b_1b_2. \end{aligned}$$

In Exercise 2.5.3 we have seen that

$$\begin{aligned} &c_t(E\otimes E') \\ &= \prod_{i,j}(1 + (a_i + b_j)t) \\ &= (1 + (a_1 + b_1)t)\,(1 + (a_1 + b_2)t)\,(1 + (a_2 + b_1)t)\,(1 + (a_2 + b_2)t) \\ &= [1 + t\,(2a_1 + b_1 + b_2) + t^2\left(a_1^2 + a_1b_2 + a_1b_1 + b_1b_2\right)] \\ &\qquad\qquad \cdot [1 + t\,(2a_2 + b_2 + b_1) + t^2\left(a_2^2 + a_2b_2 + a_2b_1 + b_1b_2\right)]. \end{aligned}$$

- In t we get:

$$t(2a_1 + b_1 + b_2 + 2a_2 + b_2 + b_1) = t[2(a_1 + a_2) + 2(b_1 + b_2)]$$
$$\Rightarrow c_1(E \otimes E') = 2c_1(E) + 2c_1(E').$$

- In t^2 we get:

$$\begin{aligned}
&t^2[a_1^2 + a_1b_2 + a_1b_1 + b_1b_2 + a_2^2 + a_2b_2 + a_2b_1 + b_1b_2 \\
&\qquad + (2a_1 + b_1 + b_2)(2a_2 + b_2 + b_1)] \\
&= t^2[a_1^2 + a_1(b_1 + b_2) + 2b_1b_2 + a_2^2 + a_2(b_1 + b_2) \\
&\qquad + 4a_1a_2 + 2a_1b_2 + 2a_1b_1 + 2a_2b_1 + b_1b_2 + b_1^2 + 2a_2b_2 + b_2^2 + b_1b_2] \\
&= t^2[a_1^2 + a_2^2 + b_1^2 + b_2^2 + 4b_1b_2 + 4a_1a_2 + a_1(b_1 + b_2) + a_2(b_1 + b_2) \\
&\qquad + 2a_1(b_1 + b_2) + 2a_2(b_1 + b_2)] \\
&= t^2[a_1^2 + a_2^2 + b_1^2 + b_2^2 + 4b_1b_2 + 4a_1a_2 + (b_1 + b_2)(3a_1 + 3a_2)] \\
&= t^2[(a_1 + a_2)^2 - 2a_1a_2 + (b_1 + b_2)^2 - 2b_1b_2 + 4b_1b_2 + 4a_1a_2 \\
&\qquad + (b_1 + b_2)(3(a_1 + a_2))] \\
&= t^2[(a_1 + a_2)^2 + (b_1 + b_2)^2 + 2a_1a_2 + 2b_1b_2 + (b_1 + b_2)(3(a_1 + a_2))
\end{aligned}$$

$$\begin{aligned}
\Rightarrow c_2(E \otimes E') &= c_1(E)^2 + c_1(E')^2 + 2c_2(E) + 2c_2(E') + c_1(E')(3c_1(E)) \\
&= c_1(E)^2 + c_1(E')^2 + 3c_1(E)c_1(E') + 2c_2(E) + 2c_2(E').
\end{aligned}$$

Remark 2.5.20. Since we are able to express each term of the Chern character of a vector bundle in terms of its Chern classes (cf. Proposition 2.5.8), this allows us to deduce formulas for the Chern classes of tensor products for any degree by using the same approach as in Exercise 2.5.18 where we did the computations for the first 2 Chern classes. The interested reader may find more details about this in [11].

2.5.2 Todd Class

Definition 2.5.21 (**Todd Class**). Let

$$c_t(E) = \prod_{i=1}^{r}(1 + a_it),$$

where the a_i's are the Chern roots of E. Then we define the **Todd class** of E by

$$\mathrm{td}(E) = \prod_{i=1}^{r} \frac{a_i}{1 - e^{-a_i}},$$

where

$$\frac{x}{1-e^{-x}} = \sum_{i=0}^{\infty} \frac{(-1)^i B_i}{i!} x^i = 1 + \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \cdots$$

and the B_i's are the Bernoulli numbers.

Remark 2.5.22 (Bernoulli number). The Bernoulli numbers are given by the following recursive definition:

$$\begin{cases} B_i &= -\sum_{k=0}^{i-1} \binom{i}{k} \frac{B_k}{i-k+1} \\ B_0 &= 1. \end{cases}$$

The first few Bernoulli numbers are:

B_0	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}
1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$

In the way it is defined here, the numbers B_i are known as the **first Bernoulli numbers**. One can also use the convention that $B_1 := \frac{1}{2}$. The numbers B_i are then known as the **second Bernoulli numbers**. However, except for this one difference, the first and second Bernoulli numbers agree.
This allows us to conclude that

$$\begin{aligned} \frac{x}{1-e^{-x}} &= \sum_{i=0}^{\infty} \frac{(-1)^i B_i}{i!} x^i \\ &= 1 + \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \frac{1}{30240}x^6 - \frac{1}{1209600}x^8 + \cdots . \end{aligned}$$

Remark 2.5.23. Again, as it was already the case for the Chern character, the Todd class is a symmetric expression in the a_i's, so by applying the splitting principle it can be expressed as a polynomial in the $c_i(E)$, with rational coefficients. Below, we will give the first five terms of this polynomial. The computations are detailed in Exercise 2.5.25.

$$\begin{aligned} \mathrm{td}(E) = 1 \,+\, & \frac{1}{2}c_1 \,+\, \frac{1}{12}\,(\,c_1^2 \,+\, c_2\,) \,+\, \frac{1}{24}c_1c_2 \\ & -\, \frac{1}{720}\,(\,c_1^4 \,-\, 4c_1^2c_2 \,-\, 3c_2^2 \,-\, c_1c_3 \,+\, c_4\,) \,+\, \cdots \end{aligned}$$

where we set $c_i = c_i(E)$, $c_i = 0$ if $i > r$.
Again, from this interpretation one easily concludes that $\mathrm{td}(E) \in H^*(X, \mathbb{Z}) \otimes \mathbb{Q}$.

Exercise 2.5.24. Let us check the equality of the preceding Remark for some vector bundle E of rank 2.
Therefore, imagine for example that $E = L_1 \oplus L_2$, where L_1 and L_2 are two line bundles on X.
In Example 2.5.6 we have already explained that $c_1(E) = a_1 + a_2$ and $c_2(E) = a_1 \cdot a_2$.
By definition we have that

$$\begin{aligned}
&\mathrm{td}(E)\\
&= \left(\frac{a_1}{1-e^{-a_1}}\right)\cdot\left(\frac{a_2}{1-e^{-a_2}}\right)\\
&= \left(1+\frac{1}{2}a_1+\frac{1}{12}a_1^2-\frac{1}{720}a_1^4+\cdots\right)\cdot\left(1+\frac{1}{2}a_2+\frac{1}{12}a_2^2-\frac{1}{720}a_2^4+\cdots\right)\\
&= 1+\frac{1}{2}a_1+\frac{1}{2}a_2+\frac{1}{12}a_1^2+\frac{1}{12}a_2^2+\frac{1}{4}a_1a_2+\frac{1}{24}a_1^2a_2+\frac{1}{24}a_1a_2^2\\
&-\frac{1}{720}a_1^4-\frac{1}{720}a_2^4+\frac{1}{144}a_1^2a_2^2+\cdots\\
&= 1+\frac{1}{2}(a_1+a_2)+\frac{1}{12}(a_1^2+a_2^2+3a_1a_2)+\frac{1}{24}(a_1^2a_2+a_1a_2^2)\\
&-\frac{1}{720}(a_1^4+a_2^4-5a_1^2a_2^2)+\cdots.
\end{aligned}$$

Now, we already know that $a_1 + a_2 = c_1(E)$. Moreover, we have that

$$a_1^2+a_2^2+3a_1a_2 = (a_1+a_2)^2+a_1a_2 = c_1(E)^2+c_2(E),$$

$$a_1^2a_2+a_1a_2^2 = a_1a_2(a_1+a_2) = c_2(E)c_1(E)$$

and

$$\begin{aligned}
a_1^4+a_2^4-5a_1^2a_2^2 &= (a_1+a_2)^4-4a_1^3a_2-4a_1a_2^3-11a_1^2a_2^2\\
&= (a_1+a_2)^4-4a_1a_2(a_1^2+a_2^2)-11(a_1a_2)^2\\
&= c_1(E)^4-4c_2(E)(c_1(E)^2-2c_2(E))-11c_2(E)^2\\
&= c_1(E)^4-4c_2(E)c_1(E)^2+8c_2(E)^2-11c_2(E)^2\\
&= c_1(E)^4-4c_2(E)c_1(E)^2-3c_2(E)^2.
\end{aligned}$$

This allows us to conclude that

$$\begin{aligned}\mathrm{td}(E) &= 1+\frac{1}{2}(a_1+a_2)+\frac{1}{12}(a_1^2+a_2^2+3a_1a_2)+\frac{1}{24}(a_1^2a_2+a_1a_2^2)\\ &\quad-\frac{1}{720}(a_1^4+a_2^4-5a_1^2a_2^2)+\cdots\\ &= 1+\frac{1}{2}c_1(E)+\frac{1}{12}(c_1(E)^2+c_2(E))+\frac{1}{24}(c_1(E)c_2(E))\\ &\quad-\frac{1}{720}(c_1(E)^4-4c_1(E)^2c_2(E)-3c_2(E)^2)+\cdots\end{aligned}$$

i.e. we end up with the equality given in Remark 2.5.23.

Exercise 2.5.25. As we already did for the Chern character, let us check the equality for the Todd class given in Remark 2.5.23 for the general case, and let us even try to compute the next term of it. Therefore, let E be an arbitrary vector bundle of rank n and $a_1,\ldots,a_n$ be the corresponding Chern roots, *i.e.*

$$\begin{aligned}c_1 &= \sum_i a_i,\\ c_2 &= \sum_{i<j} a_ia_j,\\ c_3 &= \sum_{i<j<k} a_ia_ja_k,\\ &\vdots\\ c_n &= \sum_{i_1<i_2<\cdots<i_n} a_{i_1}a_{i_2}\cdots a_{i_n}.\end{aligned}$$

By definition, we have that

$$\begin{aligned}&\mathrm{td}(E)\\ &=\prod_{i=1}^{n}\frac{a_i}{1-e^{-a_i}}\\ &=\prod_{i=1}^{n}\left(1+\frac{1}{2}a_i+\frac{1}{12}a_i^2-\frac{1}{720}a_i^4+\frac{1}{30240}a_i^6+\cdots\right).\end{aligned}$$

Let us compute separately the different terms which depend on the degree of the involved a_i's:

- Degree 0

$$1^n\boxed{=1}$$

- Degree 1

$$\frac{1}{2}\sum_i a_i \boxed{= \frac{1}{2}c_1}$$

- Degree 2

$$\begin{aligned}\frac{1}{4}\sum_{i<j} a_i a_j + \frac{1}{12}\sum_i a_i^2 &= \frac{1}{4}c_2 + \frac{1}{12}\Sigma_2 \\ &= \frac{1}{4}c_2 + \frac{1}{12}\left(c_1^2 - 2c_2\right) \\ &= \frac{1}{4}c_2 + \frac{1}{12}c_1^2 - \frac{1}{6}c_2 \\ &\boxed{= \frac{1}{12}(c_1^2 + c_2)}\end{aligned}$$

- Degree 3

$$\begin{aligned}\frac{1}{8}\sum_{i<j<k} a_i a_j a_k + \frac{1}{24}\sum_{i\neq j} a_i^2 a_j &= \frac{1}{8}c_3 + \frac{1}{24}\sum_{i<j} a_i a_j (a_i + a_j) \\ &= \frac{1}{8}c_3 + \frac{1}{24}\sum_{i<j} a_i a_j \left(c_1 - \sum_{\mu\neq i,j} a_\mu\right) \\ &= \frac{1}{8}c_3 + \frac{1}{24}\left(c_1 c_2 - 3\sum_{i<j<k} a_i a_j a_k\right) \\ &= \frac{1}{8}c_3 + \frac{1}{24}c_1 c_2 - \frac{1}{8}c_3 \\ &\boxed{= \frac{1}{24}c_1 c_2}\end{aligned}$$

- Degree 4

$$\begin{aligned}&\frac{1}{16}\sum_{i<j<k<l} a_i a_j a_k a_l + \frac{1}{144}\sum_{i<j} a_i^2 a_j^2 + \frac{1}{48}\sum_{\substack{i\neq j,k \\ j<k}} a_i^2 a_j a_k - \frac{1}{720}\sum_i a_i^4 \\ &= \frac{1}{16}c_4 + \frac{1}{144}\sum_{i<j} a_i^2 a_j^2 + \frac{1}{48}\sum_{\substack{i\neq j,k \\ j<k}} a_i^2 a_j a_k - \frac{1}{720}\Sigma_4\end{aligned}$$

From the computations of Exercise 2.5.7 we already know that

$$\sum_{i<j} a_i^2 a_j^2 = c_2^2 - 2c_1c_3 + 2c_4$$

and

$$\sum_{\substack{i\neq j,k \\ j<k}} a_i^2 a_j a_k = c_1c_3 - 4c_4,$$

so that finally,

$$\begin{aligned}
&\frac{1}{16}c_4 + \frac{1}{144}\sum_{i<j} a_i^2 a_j^2 + \frac{1}{48}\sum_{\substack{i\neq j,k \\ j<k}} a_i^2 a_j a_k - \frac{1}{720}\Sigma_4 \\
&= \frac{1}{16}c_4 + \frac{1}{144}(c_2^2 - 2c_1c_3 + 2c_4) + \frac{1}{48}(c_1c_3 - 4c_4) \\
&\qquad - \frac{1}{720}(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4) \\
&= \frac{1}{720}(45c_4 + 5c_2^2 - 10c_1c_3 + 10c_4 + 15c_1c_3 - 60c_4 \\
&\qquad - c_1^4 + 4c_1^2c_2 - 4c_1c_3 - 2c_2^2 + 4c_4) \\
&= \frac{1}{720}\left(-c_1^4 + 4c_1^2c_2 + 3c_2^2 + c_1c_3 - c_4\right) \\
&\boxed{= -\frac{1}{720}\left(c_1^4 - 4c_1^2c_2 - 3c_2^2 - c_1c_3 + c_4\right)}
\end{aligned}$$

- Degree 5

$$\begin{aligned}
&\frac{1}{32}\sum_{i<j<k<l<m} a_ia_ja_ka_la_m - \frac{1}{1440}\sum_{i\neq j} a_i^4 a_j + \frac{1}{288}\sum_{\substack{k\neq i,j \\ i<j}} a_i^2 a_j^2 a_k \\
&\qquad + \frac{1}{96}\sum_{\substack{i\neq j,k,l \\ j<k<l}} a_i^2 a_j a_k a_l \\
&= \frac{1}{32}c_5 - \frac{1}{1440}\sum_{i\neq j} a_i^4 a_j + \frac{1}{288}\sum_{\substack{k\neq i,j \\ i<j}} a_i^2 a_j^2 a_k + \frac{1}{96}\sum_{\substack{i\neq j,k,l \\ j<k<l}} a_i^2 a_j a_k a_l
\end{aligned}$$

Again, from the computations of Exercise 2.5.7 we already know that

$$\sum_{i\neq j} a_i^4 a_j = c_1^3c_2 - 3c_1c_2^2 + 5c_2c_3 - c_1^2c_3 + c_1c_4 - 5c_5,$$

$$\sum_{\substack{k\neq i,j\\ i<j}} a_i^2 a_j^2 a_k = c_2c_3 - 3c_1c_4 + 5c_5$$

and

$$\sum_{\substack{i\neq j,k,l\\ j<k<l}} a_i^2 a_j a_k a_l = c_1c_4 - 5c_5,$$

so that finally,

$$\begin{aligned}
&= \frac{1}{32}c_5 - \frac{1}{1440}\sum_{i\neq j} a_i^4 a_j + \frac{1}{288}\sum_{\substack{k\neq i,j\\ i<j}} a_i^2 a_j^2 a_k + \frac{1}{96}\sum_{\substack{i\neq j,k,l\\ j<k<l}} a_i^2 a_j a_k a_l \\
&= \frac{1}{1440}(45c_5 - c_1^3c_2 + 3c_1c_2^2 - 5c_2c_3 + c_1^2c_3 - c_1c_4 + 5c_5 \\
&\qquad\qquad + 5c_2c_3 - 15c_1c_4 + 25c_5 + 15c_1c_4 - 75c_5) \\
&\boxed{= \frac{1}{1440}(3c_1c_2^2 - c_1^3c_2 + c_1^2c_3 - c_1c_4)}
\end{aligned}$$

Example 2.5.26. Let L_1 and L_2 be two line bundles on X. Then using the equality given in Remark 2.5.23, one concludes that

$$\mathrm{td}(L_1) = 1 + \frac{1}{2}c_1(L_1) + \frac{1}{12}c_1(L_1)^2 - \frac{1}{720}c_1(L_1)^4 + \cdots$$

and

$$\mathrm{td}(L_2) = 1 + \frac{1}{2}c_1(L_2) + \frac{1}{12}c_1(L_2)^2 - \frac{1}{720}c_1(L_2)^4 + \cdots .$$

If we multiply both expressions we get that

$$\begin{aligned}
&\mathrm{td}(L_1)\cdot\mathrm{td}(L_2) \\
&= 1 + \frac{1}{2}\left(c_1(L_1) + c_1(L_2)\right) + \frac{1}{12}\left(c_1(L_1)^2 + c_1(L_2)^2\right) \\
&\quad - \frac{1}{720}\left(c_1(L_1)^4 + c_1(L_2)^4\right) + \frac{1}{4}c_1(L_1)c_1(L_2) \\
&\quad + \frac{1}{24}\left(c_1(L_1)^2c_1(L_2) + c_1(L_1)c_1(L_2)^2\right) + \frac{1}{144}c_1(L_1)^2c_1(L_2)^2,
\end{aligned}$$

which is nothing else than $\mathrm{td}(L_1 \oplus L_2)$ (cf. Exercise 2.5.24).
One could also get this result directly from the definition:

$$\mathrm{td}(L_1 \oplus L_2) = \left(\frac{c_1(L_1)}{1 - e^{-c_1(L_1)}}\right)\cdot\left(\frac{c_1(L_2)}{1 - e^{-c_1(L_2)}}\right) = \mathrm{td}(L_1)\cdot\mathrm{td}(L_2).$$

Theorem 2.5.27. *The Todd class is multiplicative, this means that for any two vector bundles E and E' on X we have that*

$$\mathrm{td}(E \oplus E') = \mathrm{td}(E) \cdot \mathrm{td}(E').$$

Proof. Let E be a vector bundle of rank r and E' be a vector bundle of rank s. By the splitting principle we can formally write both vector bundles as a direct sum of line bundles, *i.e.* $E = \oplus_{i=1}^{r} L_i$ and $E' = \oplus_{j=1}^{s} L'_j$ so that

$$\begin{aligned}
\mathrm{td}(E \oplus E') &= \mathrm{td}\left(\oplus_{i=1}^{r} L_i \oplus_{j=1}^{s} L'_j\right) \\
&= \prod_{i=1}^{r} \frac{c_1(L_i)}{1 - e^{-c_1(L_i)}} \prod_{j=1}^{s} \frac{c_1(L_j)}{1 - e^{-c_1(L_j)}} \\
&= \mathrm{td}(E) \cdot \mathrm{td}(E').
\end{aligned}$$

□

Remark 2.5.28. In Example 2.4.8 we have seen that Chern classes do not distinguish different extensions. This property is naturally inherited by the Todd class as it is also defined by means of Chern classes. More precisely, for a given short exact sequence $0 \to E'' \to E \to E' \to 0$ of vector bundles on X, we have that

$$\mathrm{td}(E) = \mathrm{td}(E'') \cdot \mathrm{td}(E').$$

Effectively, we already know through axiom A2 that $c(E) = c(E') \cdot c(E'')$. As Chern classes do not distinguish different extensions we also have through axiom A2 that $c(E' \oplus E'') = c(E') \cdot c(E'')$ (We considered here the short exact sequence $0 \to E' \to E' \oplus E'' \to E'' \to 0$). Combining these two equalities we get that $c(E) = c(E' \oplus E'')$, *i.e.* $c_i(E) = c_i(E' \oplus E'')$. This allows us to conclude that $\mathrm{td}(E) = \mathrm{td}(E' \oplus E'')$. However, through the preceding Theorem we know that $\mathrm{td}(E' \oplus E'') = \mathrm{td}(E') \cdot \mathrm{td}(E'')$ so that finally, $\mathrm{td}(E) = \mathrm{td}(E'') \cdot \mathrm{td}(E')$.

2.6 Applications to the n-dimensional complex projective space

2.6.1 The total Chern class of $\mathbb{CP}^n$

Definition 2.6.1. Let X be a complex manifold of dimension n. Consider its tangent bundle TX which is a complex vector bundle of rank n. We define the total Chern class of X by

$$c(X) := c(TX).$$

Theorem 2.6.2. *The total Chern class of* $\mathbb{CP}^n$ *is*

$$c(\mathbb{CP}^n) = (1+x)^{n+1}.$$

This has to be interpreted in the truncated polynomial ring (cf. Theorem 1.4.1)

$$H^*(\mathbb{CP}^n) \simeq \mathbb{Z}[x] \Big/ \left(x^{n+1}\right)$$

and allows us to conclude that

$$c_k(\mathbb{CP}^n) = \begin{pmatrix} n+1 \\ k \end{pmatrix} x^k,$$

where x *is the canonical generator of the cohomology group* $H^2(\mathbb{CP}^n, \mathbb{Z})$*, i.e.* $x = c_1(\mathscr{S}^*)$ *where* $\mathscr{S}$ *is the tautological line bundle on* $\mathbb{CP}^n$*. (cf. section 2.4)*

Proof. (Based on Proposition 7.51. in [3].) We use the exact sequence

$$0 \to \mathscr{S} \to \mathbb{CP}^n \times \mathbb{C}^{n+1} \to Q \to 0$$

of vector bundles on $\mathbb{CP}^n$ (cf. Section 2.4 and Theorem 2.5.1 for $E = \mathbb{C}^{n+1}$) and the fact that the Chern classes of a trivial bundle vanish (cf. Remark 2.4.10) to deduce

$$c(\mathscr{S}) \cdot c(Q) = 1.$$

From this it follows that

$$c(Q) = \frac{1}{1-x} = 1 + x + \cdots + x^n. \tag{1}$$

Moreover, there is a canonical isomorphism

$$T\mathbb{CP}^n \simeq \mathrm{Hom}(\mathscr{S}, Q) \simeq Q \otimes \mathscr{S}^*.$$

Using the splitting principle one can write formally $Q = L_1 \oplus \cdots \oplus L_n$ so that

$$Q \otimes \mathscr{S}^* \simeq L_1 \otimes \mathscr{S}^* \oplus \cdots \oplus L_n \otimes \mathscr{S}^*.$$

Now, let $a_i = c_1(L_i)$ be the Chern roots of Q. Then,

$$\begin{aligned}
c(\mathbb{CP}^n) &= c(Q \otimes \mathscr{S}^*) \\
&= \prod_{i=1}^{n} c(L_i \otimes \mathscr{S}^*) \\
&= \prod_{i=1}^{n} (1 + a_i + x) \\
&= \prod_{i=1}^{n} ((1 + x) + a_i) \\
&= (1+x)^n + \left(\sum_{i=1}^{n} a_i\right)(1+x)^{n-1} + \left(\sum_{i<j} a_i a_j\right)(1+x)^{n-2} + \cdots \\
&= \sum_{i=0}^{n} c_i(Q)(1+x)^{n-i} \text{ according to Example 2.4.9} \\
&= \sum_{i=0}^{n} x^i (1+x)^{n-i} \text{ follows from equality (1)} \\
&= (1+x)^{n+1} - x^{n+1} \text{ we used } a^{p+1} - b^{p+1} = (a-b)\sum_{i=0}^{p} a^{p-i} b^i \\
&= (1+x)^{n+1}.
\end{aligned}$$

□

Remark 2.6.3. Notice that $T\mathbb{CP}^n$ is a vector bundle of rank n, *i.e.* according to Remark 2.4.4 we have that $c_i(\mathbb{CP}^n) = 0, \forall i > n$. Hence, in the expression $c(\mathbb{CP}^n) = (1+x)^{n+1}$ we have to keep in mind that $x^{n+1} = 0$.

2.6.2 The Chern character and Todd class of $\mathbb{CP}^n$

Definition 2.6.4. Let X be a complex manifold of dimension n. Consider its tangent bundle TX which is a complex vector bundle of rank n. We define the Chern character of X by

$$\mathrm{ch}(X) := \mathrm{ch}(TX),$$

and analogously we define the Todd class of X by

$$\mathrm{td}(X) := \mathrm{td}(TX).$$

Example 2.6.5. Let us try to compute the Chern character of $\mathbb{CP}^n$ for $n = 1, 2, 3, 4$. Recall that by Remark 2.5.5

$$\begin{aligned}\mathrm{ch}(\mathbb{CP}^n) = n + c_1 + \frac{1}{2}\left(c_1^2 - 2c_2\right) + \frac{1}{6}\left(c_1^3 - 3c_1c_2 + 3c_3\right) \\ + \frac{1}{24}\left(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4\right) + \cdots\end{aligned}$$

and also that by Theorem 2.6.2

$$c_k(\mathbb{CP}^n) = \begin{pmatrix} n+1 \\ k \end{pmatrix} x^k.$$

- $\underline{n = 1}$
  We have that $c_1(\mathbb{CP}^1) = 2x$. Hence,

$$\begin{aligned}\mathrm{ch}(\mathbb{CP}^1) &= 1 + c_1 \\ &= 1 + 2x.\end{aligned}$$

- $\underline{n = 2}$
  We have that $c_1(\mathbb{CP}^2) = 3x$ and $c_2(\mathbb{CP}^2) = 3x^2$. Hence,

$$\begin{aligned}\mathrm{ch}(\mathbb{CP}^2) &= 2 + c_1 + \frac{1}{2}(c_1^2 - 2c_2) \\ &= 2 + 3x + \frac{1}{2}(9x^2 - 6x^2) \\ &= 2 + 3x + \frac{3}{2}x^2.\end{aligned}$$

- $\underline{n = 3}$
  We have that $c_1(\mathbb{CP}^3) = 4x$, $c_2(\mathbb{CP}^3) = 6x^2$ and $c_3(\mathbb{CP}^3) = 4x^3$. Hence,

$$\begin{aligned}\mathrm{ch}(\mathbb{CP}^3) &= 3 + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) \\ &= 3 + 4x + \frac{1}{2}(16x^2 - 12x^2) + \frac{1}{6}(64x^3 - 72x^3 + 12x^3) \\ &= 3 + 4x + 2x^2 + \frac{2}{3}x^3.\end{aligned}$$

- $\underline{n=4}$
  We have that $c_1(\mathbb{CP}^4) = 5x$, $c_2(\mathbb{CP}^4) = 10x^2$, $c_3(\mathbb{CP}^4) = 10x^3$ and $c_4(\mathbb{CP}^4) = 5x^4$. Hence,

$$\begin{aligned}\mathrm{ch}(\mathbb{CP}^4) &= 4 + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) \\ &\qquad + \frac{1}{24}(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4) \\ &= 4 + 5x + \frac{1}{2}(25x^2 - 20x^2) + \frac{1}{6}(125x^3 - 150x^3 + 30x^3) \\ &\qquad + \frac{1}{24}(625x^4 - 1000x^4 + 200x^4 + 200x^4 - 20x^4) \\ &= 4 + 5x + \frac{5}{2}x^2 + \frac{5}{6}x^3 + \frac{5}{24}x^4.\end{aligned}$$

Example 2.6.6. Let us try to compute the Todd class of $\mathbb{CP}^n$ for $n = 1, 2, 3, 4$. Recall that

$$\begin{aligned}\mathrm{td}(\mathbb{CP}^n) = 1 &+ \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 \\ &- \frac{1}{720}(c_1^4 - 4c_1^2c_2 - 3c_2^2 - c_1c_3 + c_4) + \cdots\end{aligned}$$

and also that

$$c_k(\mathbb{CP}^n) = \binom{n+1}{k} x^k.$$

- $\underline{n=1}$
  We have that $c_1(\mathbb{CP}^1) = 2x$. Hence,

$$\begin{aligned}\mathrm{td}(\mathbb{CP}^1) &= 1 + \frac{1}{2}c_1 \\ &= 1 + x.\end{aligned}$$

- $\underline{n=2}$
  We have that $c_1(\mathbb{CP}^2) = 3x$ and $c_2(\mathbb{CP}^2) = 3x^2$. Hence,

$$\begin{aligned}\mathrm{td}(\mathbb{CP}^2) &= 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) \\ &= 1 + \frac{3}{2}x + \frac{1}{12}(9x^2 + 3x^2) \\ &= 1 + \frac{3}{2}x + x^2.\end{aligned}$$

- $\underline{n=3}$
  We have that $c_1(\mathbb{CP}^3) = 4x$, $c_2(\mathbb{CP}^3) = 6x^2$ and $c_3(\mathbb{CP}^3) = 4x^3$. Hence,

$$\begin{aligned}\mathrm{td}(\mathbb{CP}^3) &= 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 \\ &= 1 + 2x + \frac{1}{12}(16x^2 + 6x^2) + \frac{1}{24}(4x \cdot 6x^2) \\ &= 1 + 2x + \frac{11}{6}x^2 + x^3.\end{aligned}$$

- $\underline{n=4}$
  We have that $c_1(\mathbb{CP}^4) = 5x$, $c_2(\mathbb{CP}^4) = 10x^2$, $c_3(\mathbb{CP}^4) = 10x^3$ and $c_4(\mathbb{CP}^4) = 5x^4$. Hence,

$$\begin{aligned}\mathrm{td}(\mathbb{CP}^4) &= 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 \\ &\qquad - \frac{1}{720}(c_1^4 - 4c_1^2c_2 - 3c_2^2 - c_1c_3 + c_4) \\ &= 1 + \frac{5}{2}x + \frac{1}{12}(25x^2 + 10x^2) + \frac{1}{24}(5x \cdot 10x^2) \\ &\qquad - \frac{1}{720}(625x^4 - 1000x^4 - 300x^4 - 50x^4 + 5x^4) \\ &= 1 + \frac{5}{2}x + \frac{35}{12}x^2 + \frac{25}{12}x^3 + x^4.\end{aligned}$$

It turns out that there exists an explicit formula giving the Todd class for an arbitrary projective space of dimension n. This is exactly the content of the next Theorem:

Theorem 2.6.7. *The Todd class of* $\mathbb{CP}^n$ *is*

$$\mathrm{td}(\mathbb{CP}^n) = \left(\frac{x}{1 - e^{-x}}\right)^{n+1},$$

where x *is the canonical generator of the cohomology group* $H^2(\mathbb{CP}^n, \mathbb{Z})$*, i.e.* $x = c_1(\mathscr{S}^*)$ *where* $\mathscr{S}$ *is the tautological line bundle on* $\mathbb{CP}^n$*. (cf. section 2.4)*

Proof. We will work here with the short exact sequence we have already used for the proof of Theorem 2.6.2 namely

$$0 \to \mathscr{S} \to I_{n+1} \to Q \to 0,$$

where $I_{n+1} = \underbrace{I \oplus \cdots \oplus I}_{\text{n+1}}$ denotes the trivial bundle on $\mathbb{CP}^n$. Tensorise this short exact sequence from the right by $\mathscr{S}^*$ in order to get

$$0 \to \underbrace{\mathscr{S} \otimes \mathscr{S}^*}_{=I} \to \underbrace{\left(\oplus_1^{n+1} I\right) \otimes \mathscr{S}^*}_{=\oplus_1^{n+1}\mathscr{S}^*} \to Q \otimes \mathscr{S}^* \to 0,$$

which is exact, since tensorising by vector bundles does not affect the exactness of a given short exact sequence. Recall that there is a canonical isomorphism

$$T\mathbb{CP}^n \simeq \mathrm{Hom}(\mathscr{S}, Q) \simeq Q \otimes \mathscr{S}^*,$$

so that we finally get the following short exact sequence:

$$0 \to I \to \oplus_1^{n+1} \mathcal{O}_{\mathbb{CP}^n}(1) \to T\mathbb{CP}^n \to 0.$$

According to Remark 2.5.28 we get that

$$\mathrm{td}\left(\oplus_1^{n+1} \mathcal{O}_{\mathbb{CP}^n}(1)\right) = \mathrm{td}(I) \cdot \mathrm{td}(\mathbb{CP}^n).$$

However, as $c_t(I) = 1$ we have that $\mathrm{td}(I) = 1$ and moreover we know that the Todd class is multiplicative (Theorem 2.5.27) so that

$$\mathrm{td}(\mathbb{CP}^n) = \mathrm{td}\left(\mathcal{O}_{\mathbb{CP}^n}(1)\right)^{n+1}.$$

Finally, we know that $c_t\left(\mathcal{O}_{\mathbb{CP}^n}(1)\right) = 1 + x \cdot t$ which allows us to conclude that

$$\mathrm{td}(\mathbb{CP}^n) = \left(\frac{x}{1 - e^{-x}}\right)^{n+1}.$$

□

Example 2.6.8. Let us try to recover the results from Example 2.6.6 by applying the formula of the preceding Theorem. We have that

$$\left(\frac{x}{1 - e^{-x}}\right)^{n+1} = \left(1 + \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \cdots\right)^{n+1}.$$

In what follows it is important to keep in mind that $x^{n+1} = 0$. (cf. Remark 2.6.3)

- $\underline{n=1}$

$$\begin{aligned}\left(\frac{x}{1-e^{-x}}\right)^2 &= \left(1+\frac{1}{2}x\right)^2 \\ &= 1+x.\end{aligned}$$

- $\underline{n=2}$

$$\begin{aligned}\left(\frac{x}{1-e^{-x}}\right)^3 &= \left(1+\frac{1}{2}x+\frac{1}{12}x^2\right)^3 \\ &= \left(1+\left(\frac{1}{2}x+\frac{1}{12}x^2\right)\right)^2 \cdot \left(1+\frac{1}{2}x+\frac{1}{12}x^2\right) \\ &= \left(1+x+\frac{5}{12}x^2\right)\cdot\left(1+\frac{1}{2}x+\frac{1}{12}x^2\right) \\ &= 1+\frac{1}{2}x+\frac{1}{12}x^2+x+\frac{1}{2}x^2+\frac{5}{12}x^2 \\ &= 1+\frac{3}{2}x+x^2.\end{aligned}$$

- $\underline{n=3}$

$$\begin{aligned}&\left(\frac{x}{1-e^{-x}}\right)^4 \\ &= \left(1+\frac{1}{2}x+\frac{1}{12}x^2\right)^4 \\ &= \left(1+x+\frac{5}{12}x^2+\frac{1}{12}x^3\right)^2 \\ &= 1+x+\frac{5}{12}x^2+\frac{1}{12}x^3+x+x^2+\frac{5}{12}x^3+\frac{5}{12}x^2+\frac{5}{12}x^3+\frac{1}{12}x^3 \\ &= 1+2x+\frac{11}{6}x^2+x^3.\end{aligned}$$

- $\underline{n=4}$

$$\left(\frac{x}{1-e^{-x}}\right)^5$$
$$= \left(\left(1+\frac{1}{2}x+\frac{1}{12}x^2-\frac{1}{720}x^4\right)^2\right)^2 \cdot \left(1+\frac{1}{2}x+\frac{1}{12}x^2-\frac{1}{720}x^4\right)$$
$$= \left(\left(1+\frac{1}{2}x\right)^2+2\left(1+\frac{1}{2}x\right)\left(\frac{1}{12}x^2-\frac{1}{720}x^4\right)+\left(\frac{1}{12}x^2-\frac{1}{720}x^4\right)^2\right)^2 \cdot \left(1+\frac{1}{2}x+\frac{1}{12}x^2-\frac{1}{720}x^4\right)$$
$$= \left(1+x+\frac{1}{4}x^2+\frac{1}{6}x^2-\frac{1}{360}x^4+\frac{1}{12}x^3+\frac{1}{144}x^4\right)^2 \cdot \left(1+\frac{1}{2}x+\frac{1}{12}x^2-\frac{1}{720}x^4\right)$$
$$= \left(1+x+\frac{5}{12}x^2+\frac{1}{12}x^3+\frac{1}{240}x^4\right)^2 \cdot \left(1+\frac{1}{2}x+\frac{1}{12}x^2-\frac{1}{720}x^4\right)$$

We have:

$$\left(1+x+\frac{5}{12}x^2+\frac{1}{12}x^3+\frac{1}{240}x^4\right)^2$$
$$= 1+x+\frac{5}{12}x^2+\frac{1}{12}x^3+\frac{1}{240}x^4+x+x^2+\frac{5}{12}x^3+\frac{1}{12}x^4+\frac{5}{12}x^2 + \frac{5}{12}x^3+\frac{25}{144}x^4+\frac{1}{12}x^3+\frac{1}{12}x^4+\frac{1}{240}x^4$$
$$= 1+2x+\frac{11}{6}x^2+x^3+\frac{251}{720}x^4,$$

so that

$$\left(\frac{x}{1-e^{-x}}\right)^5$$
$$= \left(1+2x+\frac{11}{6}x^2+x^3+\frac{251}{720}x^4\right) \cdot \left(1+\frac{1}{2}x+\frac{1}{12}x^2-\frac{1}{720}x^4\right)$$
$$= 1+\frac{1}{2}x+\frac{1}{12}x^2-\frac{1}{720}x^4+2x+x^2+\frac{1}{6}x^3+\frac{11}{6}x^2+\frac{11}{12}x^3 + \frac{11}{72}x^4+x^3+\frac{1}{2}x^4+\frac{251}{720}x^4$$
$$= 1+\frac{5}{2}x+\frac{35}{12}x^2+\frac{25}{12}x^3+x^4.$$

which agrees with the results we obtained in Example 2.6.6.

Appendix A

Category Theory

Definition A.1 (**Category**). A **category** $\mathcal{C}$ consists of a set $Ob(\mathcal{C})$ (or simply $\mathcal{C}$) of objects, and, for any two objects $A, B \in \mathcal{C}$, of a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ (or simply $\mathrm{Hom}(A, B)$) of morphisms $f : A \to B$ from A to B. Moreover, for any objects $A, B, C \in \mathcal{C}$, there exists a composition map

$$\circ : \mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \ni (f, g) \to g \circ f \in \mathrm{Hom}(A, C).$$

This operation is required to be associative and to have identities, *i.e.* for each object $A \in \mathcal{C}$ there exists an identity morphism $1_A \in \mathrm{Hom}(A, A)$ such that if $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(C, A)$, we have $f \circ 1_A = f$ and $1_A \circ g = g$ (this entails that the identities are actually unique).

The **opposite category** $\mathcal{C}^{op}$ of $\mathcal{C}$ is the category defined by the same objects $Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$ and by the "reversed morphisms" $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(A, B) := \mathrm{Hom}_{\mathcal{C}}(B, A)$. Moreover, for any objects $A, B, C \in Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$, the composition map is then given by

$$\circ : \mathrm{Hom}_{\mathcal{C}}(B, A) \times \mathrm{Hom}_{\mathcal{C}}(C, B) \ni (f, g) \to f \circ g \in \mathrm{Hom}_{\mathcal{C}}(C, A).$$

Again this operation is required to be associative and to have identities.

Remark A.2. It is important to keep in mind that a morphism between 2 objects need not necessary be a function as we are used to when working for instance in the context of groups where the objects are actually sets. The best way to look at a morphism between objects of a category which are hence not necessary sets is to characterize it as a simple arrow linking 2 objects of the considered category. Throughout this appendix, we consider that the words arrow, map and morphism are synonyms of each other.

Definition A.3 (**Additive Category**). A **category** $\mathcal{C}$ is **additive** if it satisfies the following conditions:

(i) for any $A, B \in \mathcal{C}$, $\mathrm{Hom}(A, B) \in \mathbf{Ab}$,

(ii) the composition law $\circ$ is bilinear,

(iii) there exists a zero object in $\mathcal{C}$,

(iv) the category $\mathcal{C}$ admits finite coproducts,

(v) the category $\mathcal{C}$ admits finite products.

Remark A.4. Condition (i) of the preceding definition tells us that for any objects $A, B \in \mathcal{C}$ the set of morphisms forms an abelian group, *i.e.* we can build sums and differences of morphisms. (This only holds for the set of morphisms not for the objects themselves.)
Condition (ii) yields that $\forall f, g, h \in \mathrm{Hom}(A, B)$ we have:

$$(f + g) \circ h = f \circ h + g \circ h,$$

$$f \circ (g + h) = f \circ g + f \circ h.$$

Note that $\mathrm{Hom}(A, B) \neq \emptyset$ since it is a group and for all $A \in \mathcal{C}$, $\mathrm{Hom}(A, 0) = \mathrm{Hom}(0, A) = 0$. (The morphism 0 should not be confused with the object 0.)

Notation A.5. If A and B are two objects of $\mathcal{C}$, one denotes by $A \oplus B$ their coproduct, and calls it their direct sum. One denotes as usual by $A \times B$ their product.

Example A.6.

1. If R is a ring, $Mod(R)$ is an additive category.
2. **Ban**, the category of $\mathbb{C}$-Banach spaces and linear continuous maps is additive.
3. If $\mathcal{C}$ is an additive category, then $\mathcal{C}^{op}$ is also additive and vice versa.
4. If $\mathcal{C}$ and $\mathcal{C}'$ are additive, then $\mathcal{C} \times \mathcal{C}'$ is also additive.

In what follows we will define a third kind of categories called **abelian categories**. Consider therefore an additive category $\mathcal{C}$ which admits kernels and cokernels. Let $f : A \to B$ be a morphism in $\mathcal{C}$. One defines:

$$\mathrm{Coim}\ f := \mathrm{Coker}\ h, \text{ where } h : \mathrm{Ker}\ f \to X$$
$$\mathrm{Im}\ f := \mathrm{Ker}\ k, \text{ where } k : Y \to \mathrm{Coker}\ f.$$

Consider the following diagram:

$$\begin{array}{ccccccc} \mathrm{Ker}\ f & \xrightarrow{h} & X & \xrightarrow{f} & Y & \xrightarrow{k} & \mathrm{Coker}\ f \\ & & \downarrow{\scriptstyle s} & \overset{\tilde{f}}{\nearrow} & \uparrow & & \\ & & \mathrm{Coim}\ f & \xrightarrow{u} & \mathrm{Im}\ f & & \end{array} .$$

Since $f \circ h = 0$, f factors uniquely through $\tilde{f}$, and $k \circ f$ factors through $k \circ \tilde{f}$. Since $k \circ f = k \circ \tilde{f} \circ s = 0$ and s is an epimorphism, we get that $k \circ \tilde{f} = 0$. Hence, $\tilde{f}$ factors through $\text{Ker } k = \text{Im } f$. We have thus constructed a canonical morphism:

$$\text{Coim } f \xrightarrow{u} \text{Im } f.$$

We are now ready to give the explicit definition of an **abelian category**:

Definition A.7 (**Abelian Category**)**.** Let $\mathcal{C}$ be an additive category. One says that $\mathcal{C}$ is **abelian** if:

(i) any $f \in \text{Hom}(A, B)$ admits a kernel and a cokernel,

(ii) $\forall f \in \text{Hom}(A, B)$, the natural morphism $\text{Coim } f \to \text{Im } f$ is an isomorphism.

Remark A.8. Abelian categories are the most general framework in which we are able to define exact sequences. In fact, since exact sequences are defined by means of kernels and cokernels, if we want to define this notion in the setting of categories we trivially need the existence of kernels and cokernels (condition (i)).
Moreover, the image of a morphism $f \in \text{Hom}_{\mathcal{C}}(A, B)$ can be defined by taking the kernel of the cokernel. To exemplify this fact, take a category where the objects are sets. In this context the cokernel of a morphism $f \in \text{Hom}(A, B)$ is given by $\text{Coker } f := Y/\text{Im } f$. Hence, it is clear that $\text{Ker}(\text{Coker } f) = \text{Im } f$.
Finally, at first sight condition (ii) appears not familiar at all. However, it encrypts a well-known result of Algebra and lifts it to the more general setting of abelian categories. As before consider a category where the objects are sets. Then, $\text{Coim } f := \text{Coker } h = X/\text{Ker } f$ (look at the map h as the inclusion map) and $\text{Im } f := \text{Ker } k = \text{Im } f$ (since $\text{Coker } f = Y/\text{Im } f$). Hence, we recover the fundamental isomorphism Theorem

$$X/\text{Ker } f \simeq \text{Im } f.$$

As a conclusion one can say that we just extend well-known results which appeared in the concrete case to the more abstract setting of categories.

Example A.9.

1. If R is a ring, $Mod(R)$ is an abelian category.
2. If $\mathcal{C}$ is abelian, then $\mathcal{C}^{op}$ is abelian.

Remark A.10. Notice that an exact sequence (cf. Definition 1.1.5) is defined by means of kernels. Hence, it only makes sense in the context of an abelian category.

Definition A.11 (**Functor**). Let $\mathcal{C}$ and $\mathcal{C}'$ be two categories. A **functor** $F : \mathcal{C} \to \mathcal{C}'$ is made up by a map

$$F : Ob(\mathcal{C}) \ni A \mapsto F(A) \in Ob(\mathcal{C}')$$

and, for any $A, B \in Ob(\mathcal{C})$, by a map

$$F : \mathrm{Hom}_{\mathcal{C}}(A, B) \ni f \mapsto F(f) \in \mathrm{Hom}_{\mathcal{C}'}(F(A), F(B)).$$

In addition, these assignments have to respect the categorical structure, *i.e.* the composition and the identities. More precisely, we ask that $F(g \circ f) = F(g) \circ F(f)$ and that $F(1_A) = 1_{F(A)}$, for all composable f and g and for all A.

Definition A.12 (**Contravariant functor**). Let $\mathcal{C}$ and $\mathcal{C}'$ be two categories. A **contravariant functor** $F : \mathcal{C} \to \mathcal{C}'$ is made up by a map

$$F : Ob(\mathcal{C}) \ni A \mapsto F(A) \in Ob(\mathcal{C}')$$

and, for any $A, B \in Ob(\mathcal{C})$, by a map

$$F : \mathrm{Hom}_{\mathcal{C}}(A, B) \ni f \mapsto F(f) \in \mathrm{Hom}_{\mathcal{C}'}(F(B), F(A)).$$

In addition, these assignments have to respect the categorical structure, *i.e.* the composition and the identities. More precisely, we ask that $F(g \circ f) = F(f) \circ F(g)$ and that $F(1_A) = 1_{F(A)}$, for all composable f and g and for all A.

Remark A.13.

- Ordinary functors as in Definition A.11 are also called **covariant functors**, in order to distinguish them from contravariant ones.
- A contravariant functor from $\mathcal{C}$ to $\mathcal{C}'$ could also be described as a covariant functor going from $\mathcal{C}$ to $\mathcal{C}'^{\mathrm{op}}$.

Example A.14 (**Hom functor**). The hom-sets, *i.e.* sets of morphisms between objects give rise to important functors to the category of sets. These functors are called **hom-functors** and have numerous applications in category theory and other branches of mathematics.
For all objects $A, B \in \mathcal{C}$ we define these functors as follows:

1. $\mathrm{Hom}(A, -) : \mathcal{C} \to Set$; this is a covariant functor given by
   (a) $\mathrm{Hom}(A, -)$ maps each object X in $\mathcal{C}$ to the set of morphisms, $\mathrm{Hom}_{\mathcal{C}}(A, X)$.

(b) $\mathrm{Hom}(A,-)$ maps each morphism $f : X \to Y$ to the function

$$\mathrm{Hom}(A,-)(f) : \mathrm{Hom}_{\mathcal{C}}(A,X) \to \mathrm{Hom}_{\mathcal{C}}(A,Y)$$

given by

$$g \mapsto f \circ g \quad \forall g \in \mathrm{Hom}_{\mathcal{C}}(A,X).$$

2. $\mathrm{Hom}(-,B) : \mathcal{C} \to Set$; this is a contravariant functor given by

(a) $\mathrm{Hom}(-,B)$ maps each object X in $\mathcal{C}$ to the set of morphisms, $\mathrm{Hom}_{\mathcal{C}}(X,B)$.

(b) $\mathrm{Hom}(-,B)$ maps each morphism $h : X \to Y$ to the function

$$\mathrm{Hom}(-,B)(h) : \mathrm{Hom}_{\mathcal{C}}(Y,B) \to \mathrm{Hom}_{\mathcal{C}}(X,B)$$

given by

$$g \mapsto g \circ h \quad \forall g \in \mathrm{Hom}_{\mathcal{C}}(Y,B).$$

Definition A.15 (**Exact Functor**)**.** An exact functor is a functor that preserves exact sequences. More precisely, let $F : \mathcal{C} \to \mathcal{C}'$ be a functor and let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of objects in the category $\mathcal{C}$. We say that F is exact if $0 \to F(A) \to F(B) \to F(C) \to 0$ is exact.
One does also distinguish left exact functors from right exact functors. In the same context one says that F is left exact if $0 \to F(A) \to F(B) \to F(C)$ is exact and F is right exact if $F(A) \to F(B) \to F(C) \to 0$ is exact. (Have a look at 1.1.5 for the definition of an exact sequence.)

Definition A.16. The Hom-functors are both examples of left exact functors. (cf. Proposition 1.6.8 and Corollary 1.6.9 in [17].)

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